ADBI Working Paper Series

HUMAN CAPITAL, ECONOMIC GROWTH, AND PUBLIC EXPENDITURE

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No. 1066
January 2020

Asian Development Bank Institute

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Suggested citation:

Bethencourt, C. and F. Perera-Tallo. 2020. Human Capital, Economic Growth, and Public Expenditure. ADBI Working Paper 1066. Tokyo: Asian Development Bank Institute. Available: https://www.adb.org/publications/human-capital-economic-growth-public-expenditure

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We acknowledge the suggestions and comments by participants in the conference on Human Capital Development for Inclusive Growth and Shared Prosperity, THUIT Workshop, PET16, International Workshop on Inequality and Middle Class Development in Africa, DEGIT XVII and seminar participants at the Universidad de Vigo, TU Dortumnd, Osaka University, and UIB. We also thank Isaac Ehrlich, José-Víctor Ríos-Rull, Pierre Pestieau, Jaime Alonso-Carrera, James R. Hollyer, Ana Hidalgo-Cabrillana, Lars Kunze, Daniel Cardona and Shingo Ishiguro for helpful discussions. We are grateful to the University of Minnesota and Osaka University for their hospitality. Finally, we thank the Spanish Ministerio de Ciencia for Grant ECO2016-76818-C3-2-P for financial support.

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Abstract

To understand the weak empirical relationship between human capital and macroeconomic performance, this paper presents a model in which human capital is allocated to three activities: production, tax collection (bureaucracy), and public education. The effective tax rate is low in poor countries because tax collection requires human capital, which is scarce. Throughout the transition, the effective tax rate rises, which involves a diversion of human capital from production to bureaucracy and public education. Consequently, human capital has a weak effect on production, even when human capital is efficiently allocated. Differences in institutional quality may involve a spurious negative correlation between gross domestic product and human capital.

Keywords: economic growth, human capital, bureaucracy, public education

JEL Classification: O42, O15, D73, I2

1. Introduction

One of the most intriguing puzzles in the literature of growth and development is the role of human capital. From a theoretical point of view, the positive impact of human capital on economic growth is clear. Growth theory has recognised the contribution of human capital in the growth process since the seminal contributions of Lucas (1988) and Romer (1990). Furthermore, empirical studies have shown that the return on education is high at the micro level, especially in developing countries (see Psacharopoulos and Patrinos, 2004; Strauss and Duncan, 1995; and Psacharopoulos, 1994). However, the empirical macroeconomic literature is surprisingly mixed and ambiguous. For example, it finds not only a weak relationship between economic performance and human capital but it also shows a negative impact of human capital. Both cross-sectional studies (Kyriacou, 1991; Benhabib and Spiegel, 1994; Nonneman and Vanhoudt, 1996; and Pritchett, 2001) and more recent panel data studies (Kumar, 2006; Bond, Hoeffler and Temple, 2001; Caselli, Esquivel and Lefort, 1996; and Islam, 1995) report negative or insignificant effect of human capital on economic growth. This constitutes a puzzle that has attracted the interest of many researchers in growth and development.

A common explanation for this puzzle is the inefficient allocation of human capital to unproductive uses (see Princhett, 2001; and North, 1990), particularly in the public sector. The explanation of this paper about the weak or even negative impact of human capital on macroeconomic performance is based, as in previous contributions, on the allocation of human capital between the public and private sectors. However, in contrast with previous contributions, the explanation of this paper is not based on the inefficient allocation of human capital to unproductive activities in the public sector. The theory that this paper proposes is based on two stylized facts: (*i*) governments in developing countries have severe trouble raising public revenues and this implies a low effective tax rate (see Gordon and Li, 2009; and Easterly and Rebelo, 1993.a, 1993.b); and (*ii*) a significant portion of skilled workers in developing countries are engaged in the public sector (see Banerjee, 2006; Gelb, Knight and Sabot, 1991; Pritchett, 2001; Schmitt, 2010; and Schündeln and Playforth, 2014).

This paper presents a stylized model in which government's activities are intensive in human capital (this is consistent with stylized fact (*ii*)). More precisely, the government needs skilled workers (i.e., workers with human capital) to collect taxes (bureaucrats) and to provide teachers for the public education system. When countries are poor, human capital is scarce and because taxes are collected by skilled workers, the effective tax rate is low (this is consistent with stylize fact (*i*)). Along the transition, human capital becomes increasingly abundant, which is associated with an increasing tax rate and, consequently, an increasing deviation of human capital from production (private sector) to bureaucracy and public education (public sector). This involves a low impact of human capital on production but does not mean that human capital is involved in unproductive activities: the public education system plays a key role in the human capital formation, and tax collection (bureaucracy) is absolutely necessary to collect taxes to finance the public education system. In this sense, our paper is in line with the view that the quality of bureaucrats is important to achieve a successful development process (see Evans, 1995; Rauch and Evans, 2000; and World Bank, 1993). Furthermore, the increasing effective tax rate and the increasing abortion of human capital by the public sector are not unproductive actions, we will also show that efficient allocation behaves in this way.

Another important contribution of this paper is to analyze the key role that institutions may play in understanding the weak empirical relationship between human capital and macroeconomic performance. Countries with lower institutional quality require more bureaucrats, which increases the return of human capital and the amount of human capital at the long run. Consequently, in the long run these countries show high levels of human capital but reduced levels of GDP per capita. Thus, differences in institutions across countries may lead to a spurious negative empirical relationship between human capital and economic performance at macro level.

Our paper provides a theory of how bureaucracy; the fiscal system; and public education evolve together with economic development and how these processes interact with human capital formation. This approach is aligned with the mainstream literature of political science, which emphasizes the importance of a competent bureaucracy in the emergence of the modern state, which is essential for economic development. For example, Hollyer (2011) argues that modern bureaucracy emerges when education becomes widespread, which allows the government to hire enough qualified bureaucrats to establish a meritocratic bureaucratic system. Using a large historical dataset that includes many European countries, he found that governments are more likely to adopt modern (meritocratic) bureaucracy as education becomes widespread. Our theory, which links the development of bureaucracy with human capital, is consistent with this empirical evidence. Hollyer's (2011) results are aligned with most of the traditional literature—such as Max Weber (1978); and Gerth and Mills (1970)—and he argues that the process of rationalization and institutionalization of bureaucracy is a by-product of economic development and, therefore, economies can only grow and develop when the private sector can be certain of the impartial conduct of government functions.

We build a model in which human capital has three uses: to produce goods; to collect taxes (bureaucracy); and to produce human capital throughout the public education system (teachers). Tax collection requires skilled workers[1], bureaucrats. There is a statutory tax rate that is only implemented if the amount of bureaucrats is large enough, otherwise the effective tax rate increases with the amount of bureaucrats recruited by the government. Tax revenues are needed to pay teachers enrolled in the public education system, who play a substantial role in the formation of human capital. A feedback process arises here: a higher level of human capital implies more efficient bureaucrats, who collect more taxes that are then used to finance public expenditure on education, which in turn promotes human capital. When the starting level of per capita capital is smaller than the steady state value, human capital is growing throughout the transition to the steady state. Tax revenues and skilled workers devoted to public education also increase along with the transition. With regard to the effective tax rate, there are two stages of "fiscal development". When human capital is very scarce and expensive, the government does not recruit enough bureaucrats to implement the effective tax rate. In the transition, the effective tax rate and the amount of bureaucrats increases, while the proportion of skilled workers devoted to the private sector (to produce goods) declines. A second stage of fiscal development starts when the amount of human capital reaches a certain threshold level. The bureaucracy reaches the point at which the effective tax rate coincides with the statutory tax rate. After this level, the tax rate and the bureaucracy sector remains stationary and any increment in the human capital is devoted only to the provision of education and to produce goods.

The fact that a significant part of human capital is recruited by the government during the development process diverts human capital from the private sector and may lead to a slowdown of production in the private sector. However, in contrast to the previous literature, the increasing absorption of human capital by the government along the transition does not mean that human capital is inefficiently allocated. To show this, we analyze the optimal tax and public expenditure policy in the model. We find that the behavior of the efficient allocation of the model is in line with the benchmark model. When the starting level of per capita capital is smaller than the steady state level, human capital and the effective tax rate rise along with the transition. The portion of human capital devoted to the public sector increases along the transition, while the portion devoted to production (private sector) decreases. Thus, the fact that in the first stages of development an increasing part of human capital is devoted to public sector activities, such as bureaucracy and public education, is not a sign of the bad allocation of resources. On the contrary, the efficient allocation behaves exactly in this way.

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[1] We define skilled workers as those workers who have achieved skills through an education process. We define human capital as the amount of skilled workers. Thus, in our model, human capital and skilled labor are the same.

This paper stresses the important role that institutional quality plays in understanding the lack of clear empirical relationship between human capital and macroeconomic performance. We capture the institutional change as a rise in the productivity of the tax collection technology. A possible interpretation of this change is that bureaucrats more efficiently use their time because they spend less time in unproductive or rent-seeking activities. In this sense, we do not exclude the possibility of unproductive uses of human capital in the public sector that previous contributions have stressed (see Blackburn, Bose, and Haque, 2006; Mauro, 2004; and Ehrlich and Lui, 1999). Another possible interpretation is that institutions in the economy, such as firms, become more transparent and this makes it easier for bureaucrats to monitor and implement tax duties. In any case, when tax collection technology becomes more productive, the government requires fewer bureaucrats and this implies a drop in the demand for human capital. It also includes a drop in the skilled premium and in the return on human capital, which involves a reduction of human capital in the long run. Furthermore, the production sector is the beneficiary of bureaucrats who have been expelled from the public sector. Thus, both production and GDP rises in the long run with the institutional improvement. In summary, an institutional improvement reduces the per capita amount of human capital in the long run but increases the per capita GDP. Thus, if there are many countries with different degrees of institutional quality at the steady state (long run equilibrium), then those countries with better institutional quality will have less human capital but more per capita GDP. Thus, a negative spurious correlation would arise between the per capita human capital and the per capita GDP. This result sheds some light on the negative correlation between macroeconomic performance and human capital that can be found in some of the papers of the empirical literature.

This paper also empirically tests some of the implications of the model. More precisely, we show that the positive relationship between public employment and per capita income predicted by our model has solid empirical support. We also find empirical support for the predicted negative relationship between the size of bureaucracy and the institutional quality. Finally, empirical evidence is consistent with the model's prediction of the existence of a hump-shaped pattern in the share of human capital allocated in the bureaucracy during the development process.

Other papers that investigate the relevance of the allocation of human capital to understand growth include Ehrlich, Li and Liu (2017), who emphasize the role of innovative entrepreneurial as an engine of growth, and Ehrlich, Cook and Yin (2018), who emphasize the importance of the quality of higher education. These two papers offer new channels to empirically test the relationship between human capital and economic growth. The second contribution is especially related to our paper because it stresses the role of institutional change; the quality of education system; and the importance of public education to generate growth through human capital accumulation. Our paper adopts these ideas but focusing on the way in which public education is financed, and the feedback relationship between the public education and the fiscal system.

This paper proceeds as follows. Section 2 presents a model where human capital is used to produce: goods; education (teachers); and public revenues (bureaucrats). Section 3 analyses the behavior and decisions of agents in the economy. Section 4 shows the resulting allocation of human capital among the three sectors. Section 5 characterizes the steady state of the economy. Section 6 describes the dynamics of the economy. Section 7 shows the results of several experiments. Section 8 analyzes the optimal allocation of the economy. Section 9 reports empirical evidence supporting main findings and, finally, our conclusions are presented. All of the proofs are included in the Appendix.

2. A three-sector model

Time is continuous and endless, and it is indexed by $t \in \Re_{+}$. There are two factors of production[2]: human capital (or skilled labor) and raw (unskilled) labor. There is a continuum of workers who are either skilled (workers with human capital) or unskilled, while each type of worker has one unit of her type of labor. We define skilled workers as those workers that have reached skills through the education process (which we will explain later on). We define human capital as the amount of skilled workers. Thus, in our model, human capital and skilled labor are the same. The per capita amount of human capital (or per capita amount of skilled workers) is denoted as h. It follows from this assumption that each worker has one unit of his type of labor and that the per capita amount of unskilled labor is equal to $1-h$.

The fertility rate is constant and denoted by $b > 0$. Agents survive to the next period with probability $1-m$, where $m \in (0,1)$ is the mortality rate. This implies that population increases at a constant rate $n \equiv b - m \geq 0$

There are three sectors in the economy:

- Production of consumption goods: this uses human capital and unskilled labor. The per capita amount of human capital devoted to production is denoted by h_y, whereas

[2] We want to focus our attention on the dynamics of human capital throughout the transition to the steady state equilibrium and, especially, on the human capital reallocation among the different sectors of the economy. Consequently, we simplify the model adopting the assumption that the unique reproducible factor is the human capital. This simplification assumption is justified because the introduction of another reproducible factor would not alter the reallocation mechanisms of the human capital along the transition.

the per capita amount of unskilled labor devoted to production is denoted by l.

- Production of human capital (education system): the agents are born unskilled, if they want to become skilled workers they have to be involved in an education process. The production of human capital requires human capital and unskilled labor. The skilled workers involved in the education system will be called teachers. The per capita amount of skilled workers devoted to production of human capital (teachers) is denoted by h_h. The education system also requires unskilled workers: unskilled workers who become skilled workers in the future. These unskilled workers are called students. The per capita amount of students are denoted by s. Education is provided by the government; that is, the education system is public and financed by taxes. This means that teachers are recruited by the government.

- Production of tax revenues (tax collection): collecting taxes is costly but necessary because taxes finance the public education system. The collection of taxes requires human capital. Skilled workers recruited by the government to collect taxes are called bureaucrats and the per capita amount of them is denoted by h_b.

In summary, there are two factors: human capital and unskilled labor. Human capital may be used to produce goods, h_y, to produce human capital (teachers), h_h, or to collect taxes (bureaucrats), h_b. While unskilled labor may be used to produce goods, l, or to produce human capital (students), s.

2.1. Production of consumption goods

A consumption good is produced according to a Cobb-Douglas production function:

$$y(t) = Al(t)^{1-\alpha} h_y(t)^{\alpha} \tag{1}$$

where $y(t)$ denotes per capita production of goods; $h_y(t)$ the per capita human capital dedicated the production of goods; $l(t)$ the per capita unskilled labor dedicated to the production of goods at t; parameter $A \in R_{++}$ is the total factor productivity; and $\alpha \in (0,1)$ is the human capital share.

2.2. Production of human capital (education system)

Agents are born unskilled, if they want to become skilled workers they have to be involved in an education process which is costly. Individuals have to devote their whole time to education during one period. We will call the agent being educated a student and we will denote the per capita amount of students as s. Furthermore, a student reaches human capital and becomes a skilled worker with probability $\mu(h_h/s)$, which depends on the ratio teachers-student h_h/s. The teachers are supplied by the government. Thus, the amount of skilled workers behaves according to the following law of motion:

$$\dot{H}(t) = \mu\left(\frac{h_h(t)}{s(t)}\right) S(t) - mH(t)$$

where $S(t)$ is the total amount of students. This equation shows that the total amount of skilled workers, $H(t)$, increases with the amount of unskilled workers that acquire human capital through the education process, $\mu(h_h(t)/s(t))\, S(t)$ and decreases with the number of skilled workers that die, $mH(t)$. If we rewrite this equation in per capita terms we get:

$$\dot{h}(t) = \mu\left(\frac{h_h(t)}{s(t)}\right) s(t) - bh(t) \tag{2}$$

The probability that a student becomes a skilled worker is as follows:

$$\mu\left(\frac{h_h(t)}{s(t)}\right) = \begin{cases} \left(\frac{h_h(t)}{s(t)}\right)^{\xi} & \text{if } \left(\frac{h_h(t)}{s(t)}\right) \leq 1 \\ 1 & \text{if } \left(\frac{h_h(t)}{s(t)}\right) > 1 \end{cases}$$

Note that the probability of effectively reaching human capital, $\mu(.)$, decreases in the parameter ξ, and that when $\xi = 0$, then this probability becomes one, $\mu(h_h/s) = 1$. Thus, we will consider that ξ is an inverse index of the quality of the educational system. The lower ξ, the better the performance of the public education program.

2.3. Production of public revenues (tax collection)

The government hires a certain number of skilled workers as teachers to produce human capital. A tax on human capital income is used to finance these expenditures. The government fixes a "statutory" tax rate, $\overline{\tau}$, on the earnings running from the human capital activities[3]. However, the government needs to hire bureaucrats to collect taxes. If there is no bureaucracy to manage and control the tax collection, then individuals would not pay any taxes. Thus, the effective tax rate that individuals pay depends positively on the bureaucrats

[3] In this version of the model, we assume that fiscal policy rules are fixed along time. Later on, in section 8, we will analyze the optimal fiscal policy in which fiscal policy rules are endogenous; that is, the policies developed by a benevolent social planer that maximizes social welfare (the utility of households).

that the government hires. There is a technology that translates the bureaucracy efforts in effective public revenues. In particular, the effective tax rate that is paid and which produces public revenues in period t is as follows:

$$\begin{aligned} \tau\left(h_b(t)\right) &= \begin{cases} \Gamma\left(h_b(t)\right)^{\gamma} & \text{if } h_b(t) < \overline{h}_b \\ \overline{\tau} & \text{if } h_b(t) \geq \overline{h}_b \end{cases} \qquad \overline{h}_b \equiv \left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}} \\ \Leftrightarrow \tau\left(h_b(t)\right) &= \min\left\{\Gamma\left(h_b(t)\right)^{\gamma}, \overline{\tau}\right\} \end{aligned} \tag{3}$$

where $\tau\left(h_b(t)\right)$ denotes the effective tax rate that is paid by individuals at period t, and $h_b(t)$ is the amount of per capita human capital devoted to the bureaucracy (per capita number of bureaucrats), $\Gamma > 0$ and $\gamma \in (0, 1)$. We assume that when there are more bureaucrats assigned to manage the tax collection, the effective tax rate and the amount of public revenues raised are both higher. There is a maximum number of bureaucrats, $\overline{h}_b$, which makes the effective tax rate, $\tau\left(h_b(t)\right)$, equal to the statutory tax rate, $\overline{\tau}$.

3. Agents' decisions

3.1. Households

There are many identical households, each of them with a continuum of agents of measure $N(t)$. This evolves according to the birth and the mortality rate:

$$\dot{N}(t) = bN(t) - mN(t) = (b-m)N(t) = nN(t)$$

Households are composed by skilled workers, unskilled workers and students; that is,

$$N(t) = H(t) + L(t) + S(t)$$

In per capita terms:

$$h(t) + l(t) + s(t) = 1$$

Regarding households' preferences, it is assumed that the utility function is time separable.

$$\int_0^{\infty} N(t)u\left(c(t)\right)e^{-\rho t} = N_0 \int_0^{\infty} u\left(c(t)\right)e^{-(\rho-n)t}$$

where $c(t)$ denotes the household's per capita consumption at period t, $\rho > n$ denotes the utility rate of discount and the utility function $u\left(.\right)$ is the CES utility function:

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{if } \sigma \in (0,1) \cup (1, +\infty) \\ \ln c & \text{if } \sigma = 1 \end{cases}$$

Therefore, the households' optimization problem is as follows:

$$\max_{\{c(t),s(t)\}_{t=0}^{\infty}} \int_0^{\infty} u\left(c(t)\right)e^{-(\rho-n)t}dt \tag{4}$$

$$c(t) = w_h(t)\left(1-\tau(t)\right)h(t) + w(t)\left(1-h(t)-s(t)\right) + tr(t) \tag{5}$$

$$\dot{h}\left(t\right) = \mu\left(t\right)s\left(t\right) - bh\left(t\right) \tag{6}$$

$$h\left(0\right) > 0$$

where $w(t)$ and $w_h(t)$ denote respectively the wage of the unskilled labor and the wage of skilled workers at period t. Therefore, households maximize their utility subject to: (i) their budget constraint (eq. 5), that is, the expenditure in consumption $c(t)$ should be equal to their disposable income that comes from human capital income, $w_h(t)h(t)$, and unskilled labor income, $w(t)\left(1-h(t)-s(t)\right)$, minus taxes $\tau(t)w_h(t)h\left(t\right)$ plus transfers $tr(t)$; (ii) the accumulation equation of human capital (eq. 6).

The Euler equation and the transversality condition associated to the households' optimization problem are:

$$\frac{\dot{c}\left(t\right)}{c\left(t\right)} = \frac{1}{\sigma}\left[\frac{w_h\left(t\right)\left(1-\tau(t)\right)-w\left(t\right)}{p_h(t)} + \frac{\dot{p}_h(t)}{p_h(t)} - m - \rho\right] \tag{7}$$

$$\lim_{t\to+\infty}\frac{1}{c(t)}e^{(\rho-n)t}p_h(t)h(t) = 0 \tag{8}$$

where $p_h(t) = \frac{w(t)}{\mu(t)}$ is the marginal cost of producing one unit of human capital. This cost is equal to the amount of unskilled labor required to produce one unit of human capital, $1/\mu\left(t\right)$, multiplied by the price of use of the unskilled labor (the opportunity cost), $w(t)$. The first of these conditions is the Euler equation. This equation shows that the consumption growth rate depends positively on the return of investment in human capital, $\frac{w_h(t)(1-\tau(t))-w(t)}{p_h(t)} + \frac{\dot{p}_h(t)}{p_h(t)}$, and negatively on the discount rate of the household's utility, ρ, and the "depreciation rate" of the human capital measured by the mortality rate, m. Notice that the return of the human capital takes the form of the return of an asset: the first part, $\frac{w_h(t)(1-\tau(t))-w(t)}{p_h(t)}$, captures the direct return of investment in human capital; and the second part, $\frac{\dot{p}_h(t)}{p_h(t)}$, measures the possible "capital gains" derived from changes in the price of human capital. Note that the difference with the standard case is that here individuals care about the ex ante return to human capital, $\mu(t)\frac{w_h(t)(1-\tau(t))-w(t)}{w(t)}$, instead of the ex post return, $\frac{w_h(t)(1-\tau(t))-w(t)}{w(t)}$. In other words, individuals are aware that there is a certain probability of not acquiring the human capital, $\mu(t)$, when they are inverting in it. The second equation is the standard transversality condition.

3.2. Firms in the production sector

Firms behave competitively and hire the amount of workers and human capital that maximize their profits:

$$\max_{L(t),H_y(t)} AL(t)^{1-\alpha}H_y(t)^{\alpha} - w_h(t)H_y(t) - w(t)L(t) \tag{9}$$

where $L(t)$ and $H_y(t)$ denote, respectively, the amount of unskilled labor and human capital hired by the firm at period t. The solution of this problem is:

$$\begin{aligned} \alpha A\left(\frac{L(t)}{H_y(t)}\right)^{1-\alpha} &= w_h(t) \\ (1-\alpha)\,A\left(\frac{H_y(t)}{L(t)}\right)^{\alpha} &= w(t) \end{aligned}$$

That is, firms hire a production factor until the point at which its price equals its marginal productivity. These first order conditions may be rewritten in per capita terms:

$$\alpha A\left(\frac{l(t)}{h_y(t)}\right)^{1-\alpha} = w_h(t) \tag{10}$$

$$(1-\alpha)\,A\left(\frac{h_y(t)}{l(t)}\right)^{\alpha} = w(t) \tag{11}$$

3.3. Government

Human capital is assumed to be perfectly substitutable among sectors and there is perfect competition. Thus, the wages of skilled workers are the same independently of the sector in which they work (production sector, bureaucracy or public education). The government budget constraint is as follows:

$$\tau\left(h_b\right)w_h\left(h_y + h_h + h_b\right) = w_h\left(h_h + h_b\right) + tr \tag{12}$$

The left-hand side of this expression represents the total public revenues of the government in per capita terms that come from the taxation over the human capital income. Per capita public revenues are defined by the effective tax rate multiplied by the per capita skilled workers' income. The right-hand side of this equation shows the government expenditures: (i) per capita expenditure in public education, $w_h h_h$, that is, the wages paid to teachers; (ii) per capita wages paid to bureaucrats, $w_b h_b$ and; (iii) per capita amount of transfers to households, tr, which represents all the government expenditures that are not devoted either to pay bureaucrats or teachers[4]. For simplicity, we assume that the government devotes a fraction, $\lambda \in (0,1)$, of the public revenues to hiring teachers. The remaining tax revenues are devoted to pay bureaucrats and to transfer payments to households:

$$\lambda\tau(h_b)\,w_h h = w_h h_h \tag{13}$$
$$(1-\lambda)\,\tau(h_b)\,w_h h = w_h h_b + tr \tag{14}$$

The objective of the government is to maximize net public revenues; that is, public revenues minus bureaucratic costs incurred to collect those revenues. Thus, the government hires the amount of bureaucrats that maximizes the net tax collection:

$$\max_{h_b} T(t) - w_h(t)h_b(t) \tag{15}$$

where $T(t)$ denotes the amount of public revenues (tax collection):

$$T(t) = \tau(h_b(t))\,w_h(t)h(t) = \min\left\{\Gamma\left(h_b(t)\right)^{\gamma}, \overline{\tau}\right\} w_h(t)h(t) \tag{16}$$

The solution of the problem is the optimal amount of bureaucrats:

$$h_b(h(t)) = \begin{cases} (\gamma\Gamma)^{\frac{1}{1-\gamma}}\,h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{h}_b & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{17}$$

where $\overline{h} = \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}$ and $\overline{h}_b = \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}$ denote the threshold levels of, respectively, per capita human capital and per capita bureaucrats that make the effective tax rate, $\tau(h_b)$, coincide with the statutory tax rate. Once the optimal amount of bureaucrats is obtained, it is easy to calculate the effective tax rate; per capita tax revenues; and the per capita amount of

[4] There are three good reasons to introduce other type of government expenditure in the model, besides expenditure in bureaucracy and the education system, which is represented by transfers: (*i*) it is realistic, given that empirically not all the government expenditures are devoted to pay bureaucracy and the education system; (*ii*) it simplified the analysis, to introduce transfers in the model allows us to have a simple lineal fiscal rule that relate the expenditure in the education system with public revenues; and (*iii*) it allows us to do exercises of comparative dynamic when the percentage of public resources devoted to education varies.

teachers:

$$\tau\left(h_b(h(t))\right) = \begin{cases} \left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau} & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{18}$$

$$T(h(t)) = \tau\left(h(t)\right) w_h(t)h(t) = \begin{cases} w_h(t) \left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau} w_h(t)h(t) & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{19}$$

$$h_h(h(t)) = \frac{\lambda T((h(t))}{w_h(t)} = \begin{cases} \lambda\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \lambda\overline{\tau}h(t) & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{20}$$

Finally, the transfer payments would be as follows:

$$tr(h(t), w_h(t)) = \begin{cases} \left(1-\gamma-\lambda\right) w_h(t) \left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ w_h(t)\left[(1-\lambda)\,\overline{\tau}h(t) - \overline{h}_b\right] & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{21}$$

Notice that the share of the tax collection devoted to pay bureaucrats is equal to γ, while the share devoted to pay teachers is λ. Thus, to guarantee the existence of non-negative transfer payments we assume that the fraction of taxes devoted to bureaucrats, γ, plus the fraction devoted to teachers, λ, are together equal to or smaller than one: $\gamma + \lambda \leq 1$.

4. The allocation of human capital among sectors

Once we determine the optimal amount of bureaucrats (eq. 17), h_b, and the amount of teachers (eq. 20), we obtain the amount of human capital that is dedicated to the production of goods, h_y, as the remaining amount of human capital after the two previous uses:

$$h_y(h(t)) = h(t)\text{-}h_b(h(t))\text{-}h_h(h(t)) = \begin{cases} h(t)\text{-}(\gamma+\lambda)\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ (1\text{-}\overline{\tau}\lambda)\, h(t)\text{-}\overline{h}_b & \text{if } h(t) \geq \overline{h} \end{cases} \tag{22}$$

We may also define the allocation of human capital in its three possible uses—bureaucracy (eq. 17); education (eq. 20); and production (eq. 22)—as ratios with respect to the total amount of human capital:

$$\frac{h_b(h(t))}{h(t)} = \begin{cases} \left(\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{h}_b & \text{if } h(t) \geq \overline{h} \end{cases} \tag{23}$$

$$\frac{h_h(h(t))}{h(t)} = \begin{cases} \lambda\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau}\lambda & \text{if } h(t) \geq \overline{h} \end{cases} \tag{24}$$

$$\frac{h_y(h(t))}{h(t)} = \begin{cases} 1-(\gamma+\lambda)\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ (1-\overline{\tau}\lambda) - \frac{\overline{h}_b}{h(t)} & \text{if } h(t) \geq \overline{h} \end{cases} \tag{25}$$

These ratios are displayed in Figure 1. The evolution of the three different uses of human capital depends on the evolution of the effective tax rate. As Figure 1.a displays, the effective tax rate is an increasing function of the per capita human capital until it reaches the threshold level, $\overline{h}$, in which the effective tax rate coincides with the statutory tax rate $\overline{\tau}$ (see eq. 18). Beyond this threshold, the effective tax rate is constant and equal to the statutory tax rate. When per capita human capital is low (countries are poor), collecting taxes is expensive because it requires human capital, which is scarce. Consequently, the effective tax rate is low and human capital is mostly devoted to production (see Figure 1.a and Figure 1.d). When human capital rises, it becomes less scarce and this makes the government hire more bureaucrats to implement a higher effective tax rate. Thus, the per capita amount of bureaucrats and the effective tax rate rise with human capital (see Figure 1.a and 1.b), and this allows the government to hire an increasing amount of teachers (see Figure 1.c). The increasing effective tax rate involves a reallocation of human capital from the private sector to the public sector, which implies that the portion of human capital dedicated to production declines with human capital (see Figure 1.d) while the share of bureaucrats and teachers in human capital rise with it. It may even happen that the quantity of human capital dedicated to production declines with human capital, not only as a share of human capital but also in per capita terms (this would be the case if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$, see eq. 22). This happens until the human capital reaches the threshold level $\overline{h}$ at which the effective tax rate coincides with the statutory tax rate.

Once that the statutory tax rate is reached, the tax rate is fixed independently of per capita human capital (see Figure 1.a). Thus, the government only hires the per capita amount of bureaucrats needed to collect the statutory tax rate. This implies that the share of bureaucrats in human capital decreases when per capita human capital goes up, as Figure 1.b shows. Because a constant share λ of tax revenues are dedicated to hire teachers and the tax rate is fixed at the statutory level, the share of teachers in human capital remain constant (see Figure 1.c). Given that the portion of teachers in human capital is constant but the share of bureaucrats declines with human capital, it follows that the portion of human capital dedicated to the public sector declines with human capital. Consequently, the human capital devoted to production increases with human capital, as Figure 1.d shows.

5. The definition of equilibrium

Definition 1 *Given the initial condition h_0, a competitive equilibrium is an allocation $\{c(t), s(t), h(t), h_y(t), l(t), h_b(t), h_h(t), tr(t)\}_{t=0}^{\infty}$ and a vector of prices $\{w(t), w_h(t)\}_{t=0}^{\infty}$ such that $\forall t$:*

- The solution of the households' maximization problem (4) is given by $\{c(t), s(t), h(t)\}_{t=0}^{\infty}$.

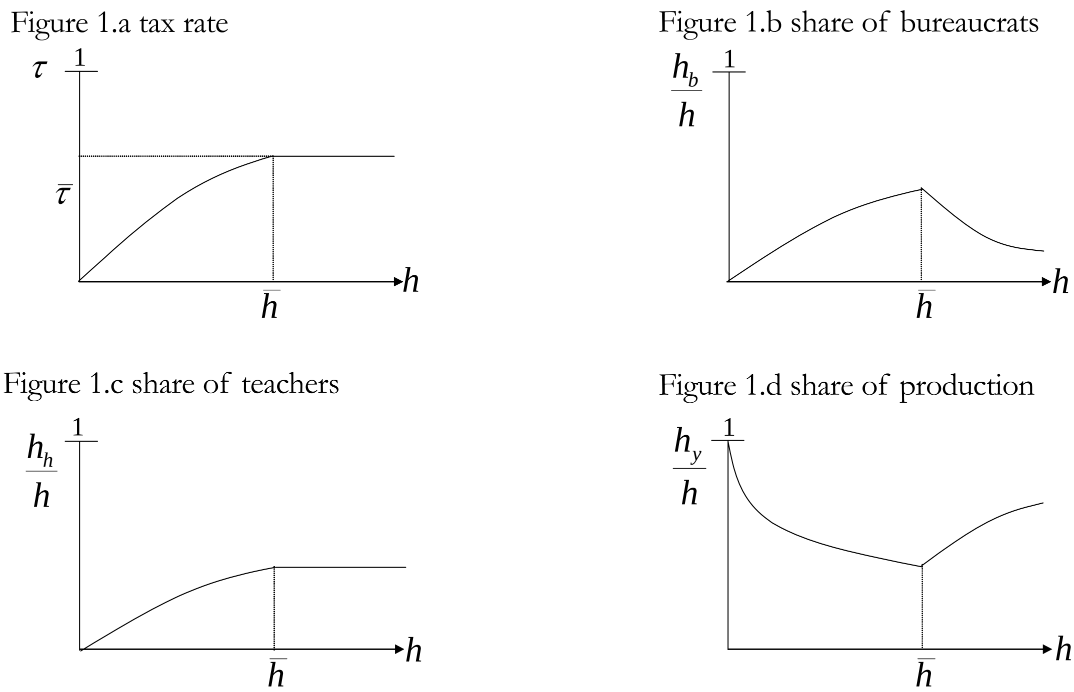

Fig. 1.— Human capital evolution among sectors

- The solution of the firm's maximization problem (9) is given by $h_y(t), l(t)$.
- The government chooses the amount of human capital devoted to bureaucracy, $h_b(t)$, which maximizes the net public revenues (eq. 15) and chooses the amount of human capital devoted to the public education system (teachers), $h_h(t)$, and the transfer payments according to fiscal policies rules (13) and (14).
- Human capital market clears: $h(t) = h_y(t) + h_h(t) + h_b(t)$.
- Unskilled labor market clears: $1 - h(t) - s(t) = l(t)$.
- Goods market clears: $y(t) = Ah_y(t)^{\alpha} l(t)^{1-\alpha} = c(t)$

Definition 2 *A steady state equilibrium is an equilibrium in which both the vector of prices and the allocation remain stable in time.*

6. Dynamic behavior

The dynamics of this economy are described by the dynamics of the consumption and the human capital variables. We now proceed to define the dynamic system of the economy.

6.1. Dynamic system

The dynamic system of this economy consists of the human capital accumulation equation (eq. 6); the Euler equation (eq. 7); and the transversality condition equation (eq. 8):

$$\begin{aligned} \dot{h}(t) &= \mu\left(\frac{h_h(h(t))}{s(t)}\right) s(t) - bh(t) \qquad (26) \\ \frac{\dot{c}(h(t), s(t))}{c(h(t), s(t))} &= \frac{1}{\sigma}\left[\frac{w_h(h(t), s(t))\,(1\text{-}\tau(h_b(h(t))))\text{-}w(h(t), s(t))}{p_h(h(t), s(t))} + \frac{\dot{p}_h(h(t), s(t))}{p_h(h(t), s(t))} - m - \rho\right] \\ \lim_{t\to+\infty} \frac{1}{(c(h(t), s(t)))} e^{-\rho t} p_h(h(t), s(t))\, h(t) &= 0 \end{aligned}$$

where $w_h(h, s)$ is the marginal product of human capital in the production sector, which coincides at equilibrium with its wage; $w(h, s)$ is the marginal product of unskilled labor in the production sector, which coincides at equilibrium with its wage; $p_h(h,s)$ is the marginal cost of producing one unit of human capital; and $c(h,s)$ is the per capita household's consumption after tax/transfers income, that is,

$$w(h,s) = (1-\alpha)A\left(\frac{h_y(h)}{1-h-s}\right)^{\alpha}; \quad w_h(h,s) = \alpha A\left(\frac{1-h-s}{h_y(h)}\right)^{1-\alpha}; \quad p_h(h,s) = \frac{w(h,s)}{\mu\left(\frac{h_h(h)}{s}\right)}$$
$$c(h,s) = w_h(h,s)\left[1-\tau(h_b(h))\right]h + w(h,s)(1-h-s) + tr(w_h(h,s),h)$$

This dynamic system may be rewritten in term of $h(t)$ and $s(t)$:

$$\dot{h}(t) = F_h(h(t), s(t)) \tag{27}$$
$$\dot{s}(t) = F_s(h(t), s(t)) \tag{28}$$

where $F_h(.)$ is the accumulation equation of human capital (26) and $F_s(.)$ is defined in the appendix.

Proposition 3 *If $\xi + \gamma \leq 1$, there exists $\overline{\Gamma} > 0$ such that if $\overline{\Gamma} < \Gamma$ then there exists an unique steady state equilibrium and $\tau^{ss} = \overline{\tau}$ ($h^{ss} > \overline{h}$).*

Note that this model includes a feedback process: more human capital involves more bureaucrats who, in turn, collect more taxes, and this allows the government to hire more teachers, which increases the return of investing in education and this promotes human capital. This feedback process might also generate a vicious circle that ends in a poverty trap: a low level of human capital; implies few bureaucrats; who collect few taxes; which involves a reduced amount of teachers; and this reduces the return on human capital; thereby discouraging human capital formation. The two key elements in this feedback process are the public revenues and the education system. Proposition 3 takes account of these two key elements: if the productivity of the education system is high enough (ξ is low enough) and the productivity of the tax collection technology is high enough (Γ is high enough and γ is low enough), then there are no poverty traps[5]. Instead, a unique steady state exists in which the effective tax rate is the statutory rate. This proposition emphasizes the importance that the quality of the education system and the institutional quality (measured by the productivity of the tax collection system) have for development

[5] As a matter of fact, if $\xi + \gamma > 1$, then the economy may converge to the trivial steady state, in which the quantity of human capital is zero. In this case, the low quality of the public education system together with the scarcity of teachers due to the low productivity of tax collection system, implies a low probability of a student become a skilled worker. This reduces the incentive to invest in education and, as a consequence, the economy is not able even to replace the skilled workers who "depreciate" (die) each period. Thus, the per capita amount of human capital declines each period converging to zero.

We will concentrate ion the case in which there exists an unique steady state, and the effective tax rate coincides with the statutory one at the steady state. Thus, we assume from now on that $\xi + \gamma \leq 1$ and $\Gamma > \overline{\Gamma}$.

Proposition 4 *If $\sigma > \overline{\sigma}$ then the steady state equilibrium is a saddle point and $s(t)$ increases when $h(t) < h^{ss}$ and decreases when $h(t) > h^{ss}$.*

The dynamic of saddle point implies that there exists a unique converging path to the steady state. In other words, given the starting level of per capita human capital, there exists only one converging equilibrium trajectory to the steady state. Figure 2 shows the phase diagram of the standard saddle point dynamics. If the starting level of per capita human capital is smaller than level at the steady state, then the number of students grows throughout the equilibrium path, converging to its steady state level. If the starting level of per capita human capital is larger than the level at the steady state, then the opposite happens.

The evolution of the number of students along transition depends greatly on the elasticity of substitution of the utility function $(1/\sigma)$. To see this, consider that the initial per capita human capital, $h(0)$, is below the level at the steady state, $h(0) < h^{ss}$; and so, due to the relative scarcity of the human capital, the return of education is high. To determine the relationship between the amount of students and human capital, we have to consider two effects: a substitution effect and a wealth effect. Insofar as countries accumulate human capital, the return of human capital decreases, which reduces the incentive to have more students in the economy. Thus, a substitution effect would imply a decrease in the number of students. Simultaneously, when countries own more human capital and can afford higher levels of consumption, then they would tend to have more students because they would like to enjoy higher levels of consumption in the future. Thus, a wealth effect would imply a rise in the amount of students. The resulting net effect would depend on the relative size of those two effects. However, the relevant case from the empirical point of view is the one in which the number of students increases during the development process. So, if we want a model that has the property of that the number of students increases during the transition while countries accumulate human capital, then the substitution effect should not be too large. This is the reason why we will concentrate on the case in which the elasticity of substitution is small enough, $\frac{1}{\sigma} < \frac{1}{\overline{\sigma}}$; that is, when the parameter sigma is large enough, $\sigma > \overline{\sigma}$.

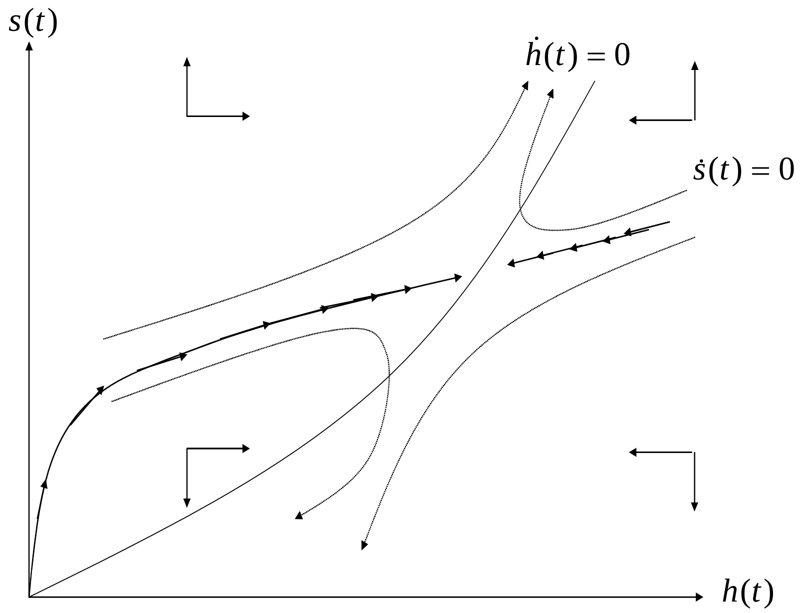

Fig. 2.— Saddle point dynamic

6.2. Microeconomic effects: the evolution of the skill premium along the transition

Empirical studies about the return of human capital at the micro level often use the skill premium as a measure of the return of human capital. The definition of skill premium should be as follows:

$$sp = \frac{w_h - w}{w}$$

In our model the skill premium defined as in the above equation would be the ex-post return of human capital. The reason for this is simple: note that the household's investment in human capital consist of investing unskilled labor. One unit of unskilled time invested in human capital has an opportunity cost that is equal to the wage of unskilled labor and a gain in return, which is an increase in the wage due to the higher wage of skilled labor. However, this gain only exists in the case of obtaining human capital, which implies that the return is ex-post. Thus, the household invests the wage of a unskilled worker, w, and it obtains the increase in wage when a worker is skilled, $w_h - w$.

It follows from eqs. (10) and (11) that:

$$sp = \frac{\alpha}{1-\alpha}\frac{1-h-s}{h_y} - 1 \tag{29}$$

The skill premium depends on the relative abundance of human capital with respect to unskilled labor in the production sector, which is the sector that determines the wage of unskilled and skilled workers. If the initial per capita human capital is below than the level at the steady state, then the unskilled labor used in production is continues to decline along with the transition because the per capita amount of skilled workers and students increases. Thus, if the per capita human capital dedicated to production increases along the transition, then the human capital becomes increasingly abundant with respect unskilled labor. Consequently, the skill premium declines along the transition. However, as we will see in next section, the per capita human capital dedicated to production does not always increase along with the transition. Nevertheless, the following proposition shows that, even in this case, the skill premium declines along the transition[6].

Proposition 5 *Assume that* $h(0) < h^{ss}$. *There is* $\widehat{\tau} > \frac{1-\gamma}{\gamma+\lambda}$ *such that, if* $\overline{\tau} \leq \widehat{\tau}$ *then* $\forall t \geq 0$ $\dot{sp}(t) < 0$.

Note that if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$, then the per capita human capital devoted to production does not always rise along with the transition (see eq. 22). Thus, according to the above proposition, the fact that the human capital devoted to production may decline along the transition does not preclude the decline of skill premium along the transition. This result is supported by the empirical evidence, which finds that the return on education declines with the per capita income level (see Psacharopoulos and Patrinos, 2004; Strauss and Duncan, 1995; and Psacharopoulos, 1994).

6.3. Macroeconomic effects: the dynamics of the allocation of human capital among sectors along the transition

We now analyze the allocation of human capital among different sectors along the transition to the steady state. If we consider that the starting per capita human capita is below the threshold $\overline{h}$, which, in turn, is smaller than the level at the steady state (see proposition 3), then we may differentiate two different stages of development along the transition: (*i*) the first stage of development, when per capita human capital is below the threshold $\overline{h}$ and consequently the effective tax rate is lower than the statutory one; and (*ii*) the second stage of development, when per capita human capital is above the threshold $\overline{h}$ and consequently

[6] If we define the skill premium taking account of the taxes (the after tax skill premium), then the skill premium would still decline along with the transition; that is, proposition 5 would hold. The tax rate increases along the transition. This feature would make the after tax skill premium decline along with the transition.

Figure 3.a tax rate

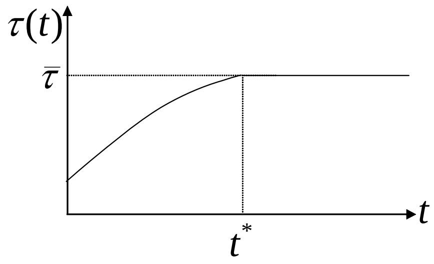

Figure 3.b share of bureaucrats

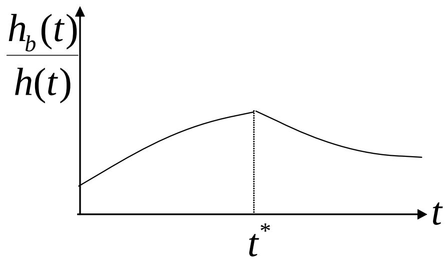

Figure 3.c share of teachers

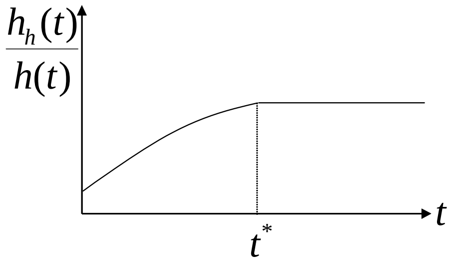

Figure 3.d share of production

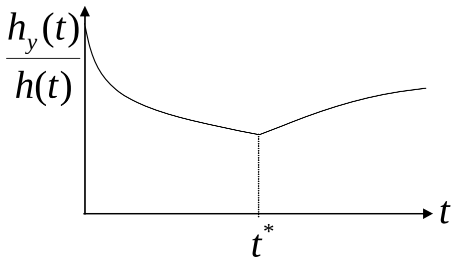

Fig. 3.— Evolution of the human capital allocation

the effective tax rate becomes the statutory one. We will define t^* as the moment at which the effective tax rate reaches the statutory tax rate (and per capita human capital reaches the threshold level $\overline{h}$). Thus, the first stage of development will take place from the initial moment of the economy until t^*, and the second stage from t^* henceforth. In the first stage of development, the scarcity of human capital precludes the government from hiring enough bureaucrats to implement the effective tax rate. This implies low levels of public revenues, which do not allow the government to hire many teachers. Because there are few bureaucrats and teachers, most human capital is dedicated to production. However, along the transition, human capital increases. The large amount of human capital allows the government to hire more bureaucrats, which increases the effective tax rate; the tax collection; and, consequently, the amount of teachers. This means that in the first stage of development, the effective tax rate, and the share of bureaucrats and teachers in human capital increase (see eqs. 18, 23, and 24, and Figures 3.a, 3.b and 3.c), whereas the share of skilled workers devoted to production is declining (see eq. 25 and Figure 3.d).

Once the statutory tax rate is reached (at moment t^*), the second stage of development starts. The tax rate is fixed at the statutory level (Figure 3.a) and the government does not need to increase the per capita amount of bureaucrats because it just need the threshold level $\overline{h}_b$ to implement the statutory tax rate. Given that the per capita amount of bureaucrats is constant and the per capita human capital rises along transition, the share of bureaucrats in human capital declines; as shown in Figure 3.b. Moreover, given that a fixed portion of public revenues is dedicated to public education and the tax rate is constant, the share of teachers in human capital is constant; as Figure 3.c. shows. Finally, because the fraction of bureaucrats in human capital falls and the fraction of teachers remains constant, the share of public sector as a whole in human capital declines. This means that the share of human capital in the private sector (i.e., production) rises in the second stage of development (after moment t^*).

Figure 4 displays the evolution of factors used in production. As we explain in Section 4, the per capita human capital devoted to production is not necessary monotonic. In fact, if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$, then the amount of per capita human capital devoted to production is not monotonic. This case is displayed in Figure 4.a. Meanwhile, because the per capita amount of human capital and the per capita amount of students rise along the transition, the level of unskilled labor falls along the transition ($l = 1 - h - s$); as Figure 4.b shows. Furthermore, the per capita amount of workers devoted to production (skilled plus unskilled workers) declines along the transition. To see this consider, the following equation:

$$h_y + l = h_y + 1 - h - s = 1 - h_b - h_h - s = \begin{cases} 1 - (\gamma + \lambda)\left(\gamma^\gamma \Gamma\right)^{\frac{1}{1-\gamma}} h^{\frac{1}{1-\gamma}} - s & \text{if } h < \overline{h} \\ 1 - \overline{h}_b - \lambda \overline{\tau} h - s & \text{if } h \geq \overline{h} \end{cases}; \quad (30)$$

This equation says that the total (per capita) amount of workers devoted to production, both skilled workers h_y and unskilled workers l, is equal to the per capita amount of workers, 1, minus the workers that do not devote their time to production: bureaucrats h_b and teachers h_h, in the skilled workers group; and students s, in the unskilled group. Because bureaucrats, teachers and students increase in the first stage of development (before t^*), the per capita quantity of workers dedicated to production declines always; as Figure 4.c shows (see eq. 30). Given that in the first stage of development the per capita amount of skilled workers devoted to production is not necessarily monotonic, and the per capita amount of unskilled workers and the per capita total amount of workers declines along transition, the production may slow down or even not be monotonic along time; as shown in Figure 4.d. Thus, the increasing drain of skilled worker from production to the public sector, together with the increasing drain of unskilled workers from production to education (the students), involves a slow down of production and this may explain the disappointing effect of human capital in economic performance at the empirical level.

In the second stage of development (after t^*), from eqs. (22) and (30) we, respectively, know that the amount of per capita human capital devoted to production increases (see

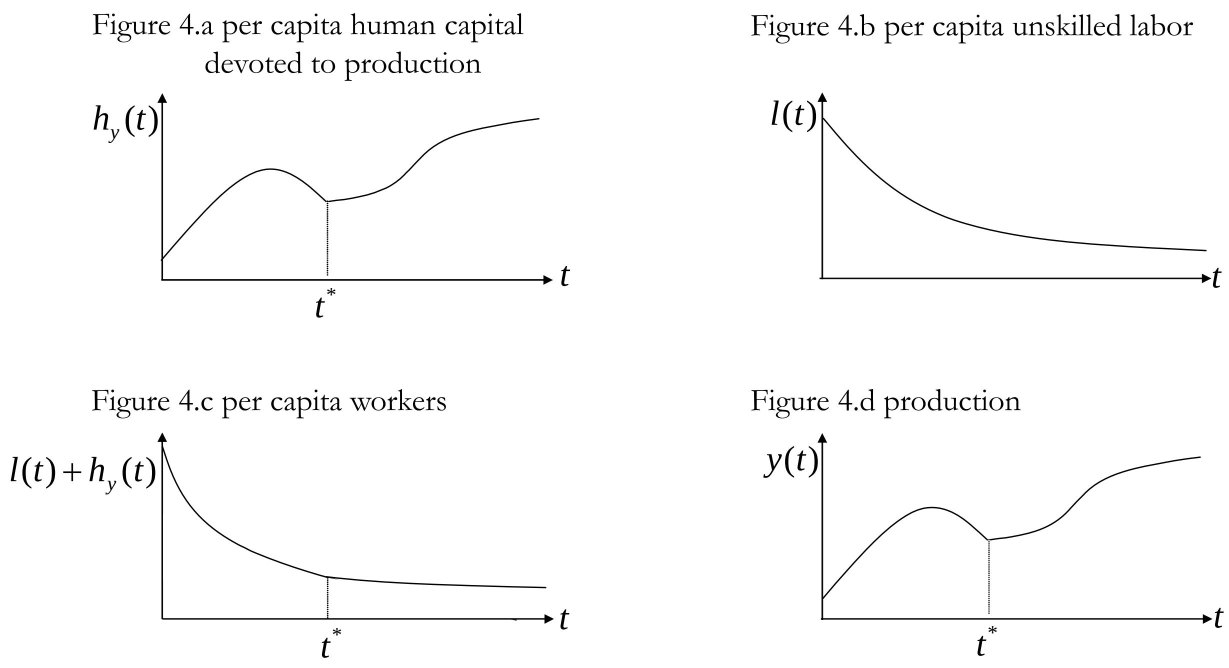

Fig. 4.— Evolution of different types of labor used in production

Figure 4.a) and the total amount of workers (Figure 4.c) decreases at a lower rate than in the first stage of development. Thus, it is plausible that production rises along the transition (Figure 4.d) which is evidenced by the empirical literature.

7. Institutional changes: the effect of an improvement in the technology of bureaucracy

In this section, we evaluate the effect of an institutional improvement through the performance of the government producing public revenues. More precisely, we analyze the effect of a technological improvement in the bureaucratic sector through an increase in the parameter Γ. In this context, a technological improvement in the bureaucracy implies that for the same amount of bureaucrats, the tax collection increases; that is, the effective tax rate is closer to the statutory tax rate. To see why this change in the tax collection technology may be interpreted as an institutional improvement, consider the following modification of the tax collection technology (see eq. 3):

$$\tau\left(h_b(t)\right) \;=\; \begin{cases} B\left[\left(1-\psi\right)h_b(t)\right]^{\gamma} & \text{if } h_b(t) < \overline{h}_b \\ \overline{\tau} & \text{if } h_b(t) \geq \overline{h}_b \end{cases} \qquad \overline{h}_b \equiv \left(\frac{\overline{\tau}}{B\left(1-\psi\right)^{\gamma}}\right)^{\frac{1}{\gamma}}$$

where $B > 0$ and $\psi \in (0,1)$ is the portion of bureaucratic time devoted to unproductive activities. If we define $\Gamma \equiv B(1-\psi)^{\gamma}$, then it is easy to see that this tax collection technology is exactly the same as the one which is already presented in eq. (3): any drop in ψ implies a improvement in the tax collection technology (a rise in Γ). Thus, a rise in Γ may be interpreted as an institutional improvement that enables bureaucrats to devote less time to unproductive activities. An alternative interpretation is that a positive institutional change may involve more transparent institutions and/or responsible taxpayers, making it easier and less costly to collect taxes.

Hence, an increase in Γ implies that a smaller amount of human capital is now required to implement the statutory tax rate. Thus, for a given amount of tax collection, the government may devote more resources to the provision of income transfers to households and a reallocation of human capital from the bureaucracy to the production sector arises. The rise in the number of skilled workers in the private sector decreases the wage of human capital, which discourages the accumulation of human capital and reduces the amount of students. Consequently, per capita human capital is smaller in the new steady state than in the initial one. Nevertheless, in spite of a lower per capita human capital at the new steady state, the per capita human capital used in production is larger in the new steady state. Furthermore, because there are fewer skilled workers and fewer students in the new steady state, there are more unskilled workers in the production sector (in per capita terms). Consequently, per capita production is higher at the new steady state. The following proposition says that not only will production rise with institutional quality (represented by the parameter Γ), but so will per capita GDP (under certain conditions); with per capita GDP, *gdp*, defined as per capita income of the country, that is,

$$gdp = wl + wh$$

Proposition 6 *If there is a technological improvement in the bureaucracy sector, measured as an increase of Γ, then the steady state levels of students and human capital decrease, and the amount of human capital dedicated to production and the production increase. Furthermore, there is a constant $\widetilde{\alpha} > 0$ such that if $\alpha < \widetilde{\alpha}$ then an increase in Γ involves an increase in gdp at the steady state.*

This proposition implies that there may be a negative relationship between institutional quality and per capita human capital, and a positive relationship between per capita GDP and institutional quality. These relationships together may generate a spurious negative relationship between per capita human capital and per capita GDP. To see this, consider many countries with different degrees of institutional quality, as represented by different levels of the parameter Γ, all of them at the steady state. Those countries with better institutions (high Γ) have higher levels of per capita production and GDP but lower levels of human capital than those with poorer institutions. Therefore, the correlation between per capita GDP and human capital would be negative.

This result does not mean that human capital does not contribute to production, it simply means that those countries with weaker institutions and consequently with lower productions are the ones that require more human capital to produce law enforcement and to encourage agents to fulfill the fiscal rules. Thus, this negative correlation between per capita GDP and human capital would be spurious and misleading in the sense that per capita human capita rises do not involve a fall in per capita GDP. In fact, the unique motor of growth in this economy is human capital. This result sheds some light on the weak or even negative correlation between economic performance and human capital that has been documented by many empirical papers (see the introduction).

The dynamics of the economy generated by an increase in Γ is described by the phase diagram in Figure 5. An increase in Γ involves a shift to the right of the locus $\dot{s} = 0$. From the starting value of per capita human capital, there exits a unique trajectory of equilibrium that converges to a steady state with a lower level of human capital and a smaller number of students but with higher level of per capita production; as proposition 6 establishes.

Figure 6 shows the evolution of the of human capital dedicated to each sector in the economy when Γ increases at period t_0. Given the fact that bureaucrats are more efficient collecting taxes, the government can implement the statutory tax rate with less bureaucrats. Thus, the government hires less bureaucrats (Figure 6.a) and spends this amount of resources on providing more transfers to households (see eqs. 17 and 21). Consequently, skilled workers are reallocated from bureaucracy to the production sector (Figure 6.c). This drop in the demand for bureaucrats reduces the wage of skilled workers; discourages the accumulation of human capital; and reduces the number of students and the per capita human capital. This downsize of human capital implies a gradual reduction in the per capita amount of teachers (Figure 6.b), which reduces the return on the human capital and further discourages the accumulation of human capital. The amount of per capita human capital devoted to production declines along the transition but it converges to a higher level than the one at the initial steady state (as we established in proposition 6 and is displayed in Figure 6.c). Finally, the reduction of students along the transition involves an increase in the per capita unskilled labor devoted to production (Figure 6.d). This last effect contributes to increase the per capita production and GDP at the new steady state.

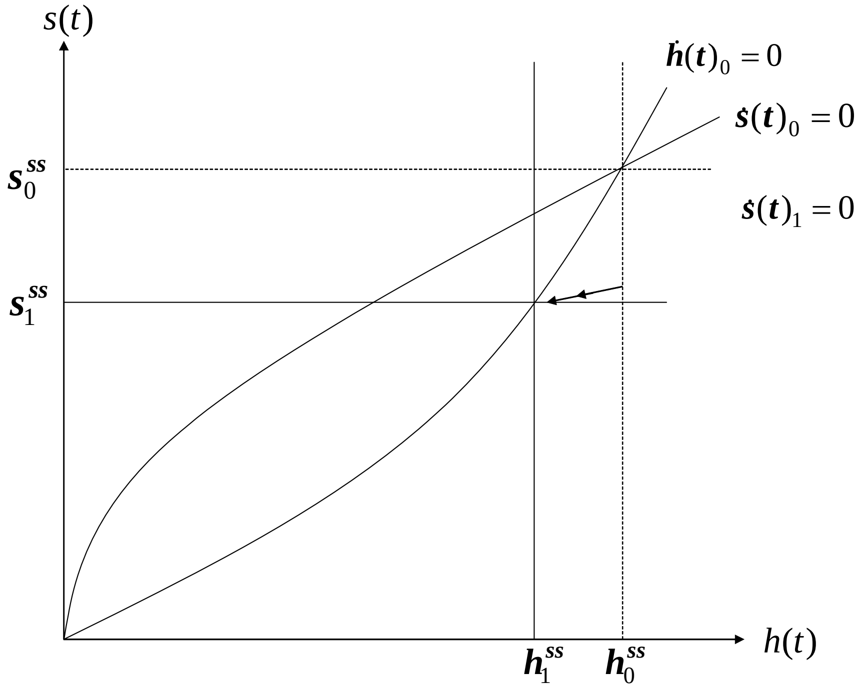

Fig. 5.— An institutional improvement

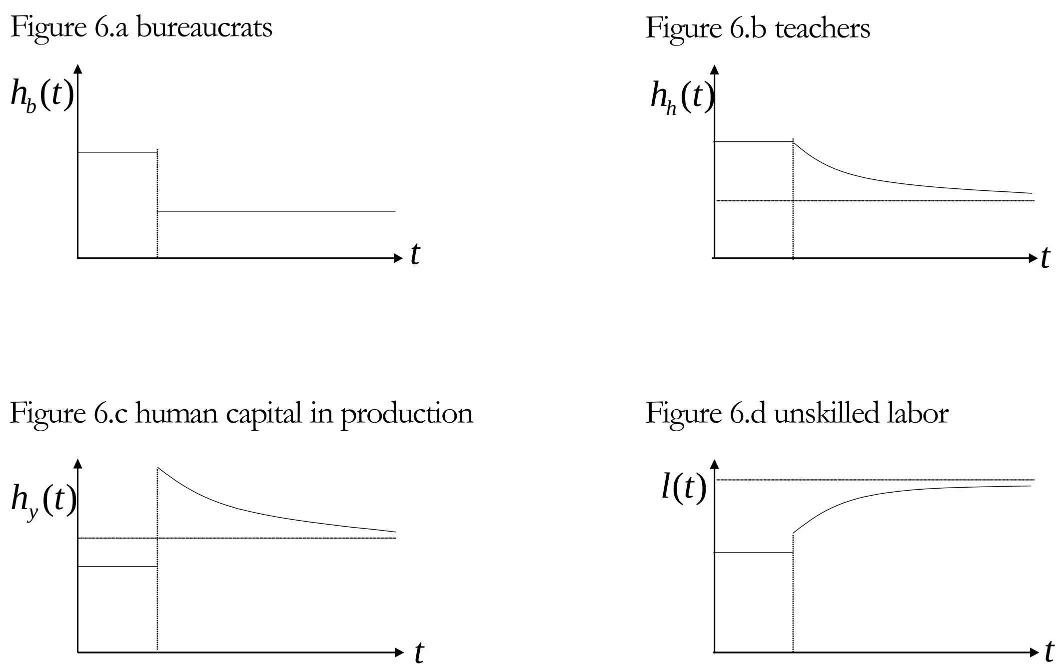

Fig. 6.— The effect of a institutional improvement

8. Optimal fiscal policy

In this section, we will analyze the tax rate that maximizes welfare in this economy. Because the statutory tax in this section is not exogenous, the (effective) tax rate is defined as follows:

$$\tau = \tau\left(h_b(t)\right) = \Gamma\left(h_b(t)\right)^{\gamma} \tag{31}$$

The budget constraint of the government is similar to the benchmark model but without transfers, which are not included in the present analysis[7]:

$$\tau\left(h_b\right) w_h h = w_h\left(h_h + h_b\right) \tag{32}$$

Using eqs. (31) and (32) together with the constraint $h = h_y + h_h + h_b$, it is possible to get the different uses of human capital in function of the effective tax rate, τ, and the per capita human capital, h:

$$h_y = h_y(h,\tau) = (1-\tau)h \tag{33}$$

$$h_b = h_b(\tau) = \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \tag{34}$$

$$h_h = h_h(h,\tau) = \tau h - \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \tag{35}$$

Because taxes are used to finance bureaucrats and teachers, a higher tax rate involve less human capital dedicated to the production of goods. Thus, the per capita human capital dedicated to production, h_y, is a decreasing function of the effective tax rate and an increasing function of per capita human capital. The per capita number of bureaucrats is a function that depends positively on the effective tax rate because a higher effective tax rate requires a larger number of bureaucrats to implement it (see eq. 31). Finally, the relationship between per capita number of teachers, h_h, and the effective tax rate shows a hump-shape form. Two offsetting mechanisms generate this hump-shape: (*i*) a higher effective tax rate increases the government revenues that finance education; and (*ii*) a larger effective tax rate raises the bureaucratic cost of implementing the effective tax rate. Substituting eqs. (33) and (35) in the production technology (eq. 1) and in equation (eq. 2) (human capital accumulation

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[7]The reason why we exclude transfers in our analysis is that transfers are clearly inefficient in this economy: if transfers are introduced, then additional bureaucrats would be required to collect necessary taxes to finance these transfers which is costly and so a well-being worsening. Given that all households are alike, they all pay the same amount of taxes and receive the same amount of transfers. Consequently, their disposable income would not change if tax collection were free. However, because tax collection is costly, the introduction of transfers would reduce the households' disposable income, thereby households not only pay taxes to finance transfers but also to finance the bureaucratic cost required to collect such as taxes.

equation), respectively, we get the per capita production of goods and the per capita human capital accumulation equation as functions of the effective tax rate:

$$\begin{aligned} y(t) &= c(t) = c(h(t), \tau(t)) = A\left((1-\tau(t))h(t)\right)^{\alpha}(1-h(t)-s(t))^{1-\alpha} \\ \dot{h}(t) &= h_h(h(t), \tau(t))^{\xi} s(t)^{1-\xi} - bh(t) \end{aligned}$$

where in the first equation we use the constraint that the per capita amount of workers is equal to one: $1 = l(t) + h(t) + s(t)$.

The benevolent social planner's problem would be as follows:

$$\begin{aligned} &\max_{\{\tau(t),s(t)\}_{t=0}^{\infty}} \int_0^{\infty} \frac{\left(A\left((1-\tau(t))h(t)\right)^{\alpha}(1-h(t)-s(t))^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t} dt \\ &\dot{h}(t) = h_h(h(t), \tau(t))^{\xi} s(t)^{1-\xi} - bh(t) \qquad (36) \\ &h(0) > 0 \end{aligned}$$

The FOCs (first order conditions) of the optimization problem (36) are as follows:

$$\frac{\alpha[c(h(t),\tau(t))]^{1-\sigma}}{1-\tau(t)} = \lambda(t)\xi\left(\frac{s(t)}{h_h(h(t),\tau(t))}\right)^{1-\xi}\left(h(t) - \frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}\right) \qquad (37)$$

$$\frac{(1-\alpha)[c(h(t),\tau(t))]^{1-\sigma}}{1-h(t)-s(t)} = \lambda(t)(1-\xi)\left(\frac{h_h(h(t),\tau(t))}{s(t)}\right)^{\xi} \qquad (38)$$

$$\frac{\left[\frac{\alpha}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right][c(h(t),\tau(t))]^{1-\sigma} + \lambda(t)\left[\xi\left(\frac{s(t)}{h_h(h(t),\tau(t))}\right)^{1-\xi}\tau(t) - m\right]}{\lambda(t)} + \frac{\dot{\lambda}(t)}{\lambda(t)} = \rho \qquad (39)$$

where $\lambda(t)$ is Lagrange's multiplier, which is interpreted as the shadow price of human capital. The first equation, FOC (37), means that the marginal cost of taxation in term of reduction of the present utility due to lower consumption, $\frac{\alpha[c(h(t),\tau(t))]^{1-\sigma}}{1-\tau(t)}$, should be equal to its marginal benefit, which is the value of the marginal increase in the future human capital due to the taxation, $\lambda(t)\xi\left(\frac{s(t)}{h_h(h,\tau)}\right)^{1-\xi}\left(h(t) - \frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}\right)$. The "marginal cost" of the taxation in the production of goods and the "marginal benefit" of the taxation in education are due to the reallocation of human capital from the production of goods (where public revenues come from) to the education sector (where the public expenditure takes place). FOC (38) means that the marginal cost of students in terms of reduction of the present utility due to lower consumption, $\frac{(1-\alpha)[c(h(t),\tau(t))]^{1-\sigma}}{1-h(t)-s(t)}$, should be equal to its marginal benefit, which consists of the marginal increase of the value of the future human capital due to students, $\lambda(t)(1-\xi)\left(\frac{h_h(h(t),\tau(t))}{s(t)}\right)^{\xi}$. The "marginal cost" of the students in the production of goods and the "marginal benefit" of the students in education is due to the reallocation of raw labor from the production of goods, to the education sector. Finally, FOC (39) means that the return of investing in human capital (left-hand side of the equation) should equates the discounted factor of the utility. The return of investing in human capital has two parts: the marginal net income of human capital divided by the price of the capital (generated in the production of final goods) and the capital gains (generated in the production of human capital). The marginal net income is the marginal increase of the utility due to the future increase of production of final goods due to human capital, $\left[\frac{\alpha}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right] [c(h(t),\tau(t))]^{1-\sigma}$, which has two components, the increase of the production due to more human capital, $\frac{\alpha}{h(t)}$, minus the reduction of the production due to the reduction in raw labor that the increase in human capital involve, $-\frac{1-\alpha}{1-h(t)-s(t)}$. To this "marginal income" in the production of final goods it should be added the value of the marginal production of future human capital in the production of future human capital, $\lambda(t)\left[\xi\left(\frac{s(t)}{\tau(t)h(t)-\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}}\right)^{1-\xi}\tau(t)\right]$, minus the value of its depreciation, $\lambda(t)m$, which consists in the value of the portion of skilled workers that die. The capital gains consist in the growth rate of the shadow price of human capital, $\frac{\dot{\lambda}(t)}{\lambda(t)}$.

Remark 7 *It follows from equation (37) that at any optimal path the marginal revenue of a higher tax rate, $h(t)$, should exceed its marginal cost (in terms of bureaucratic effort), $\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}$. That is, $h(t) > \frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}} \iff \tau(t) < (\Gamma)^{\frac{1}{1-\gamma}}(\gamma h(t))^{\frac{\gamma}{1-\gamma}}$. Thus, we will only consider tax rates that satisfied the above constraint.*

Using FOCs (37) and (38), it is possible to obtain the per capita number of students, s, as a function of per capita human capital, h, and the tax rate, τ (see the appendix):

$$s = s(h,\tau) = \frac{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau} + \frac{\xi}{1-\xi}\left(h_h(h,\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \tag{40}$$

where $h_h(h,\tau)$ is the per capita amount of teachers defined in equation (35). The per capita amount of students is a function that depends positively on the tax rate, due to two reasons: (i) a higher tax rate implies more teachers, which reduces the cost of education (increases the probability of obtaining education; and (ii) a higher tax rate involves a lower quantity of human capital in production of goods (see eq. 33), which makes human capital scarcer and better paid. The per capita amount of students is a decreasing function of the per capita human capital because abundant per capita human capital reduces its marginal productivity and the return of education. From this equation and the first order conditions defined above, it is possible to derive a dynamic system of equations as a function of per capita human capital, h, and the tax rate, τ, which is rather complicated and may be found in the appendix.

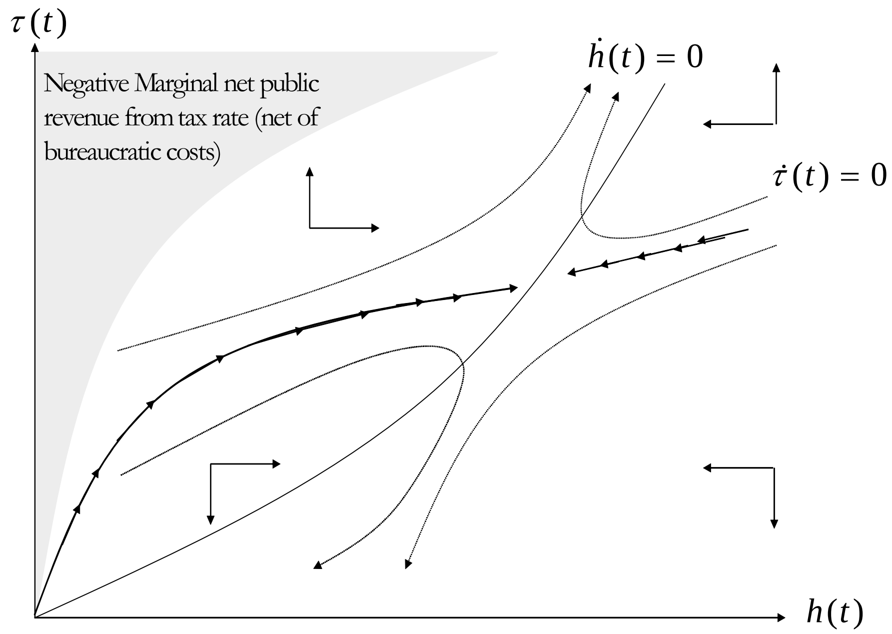

Fig. 7.— Optimal tax rate dynamic

Proposition 8 *There is a non-empty subset of the parameter space such that a steady state exists and is unique.*

Proposition 9 *If the steady state is unique, then it is a saddle point*

These propositions determine that there is a subspace of parameters in which the steady state exists, which is unique and is a saddle point. Figure 7 displays the dynamic to the steady state. The shadowed area represents combinations of human capital and tax rate in which the marginal cost of rising taxes exceed its marginal revenue; that is, the net marginal revenue from rising the tax rate is negative (net from bureaucratic cost). Thus, these combinations are never efficient (see remark 7). We see that per capital human capital and tax rate increase along the transition when per capita capital is low. This implies that per capita amount of teachers and bureaucrats increase along the transition (see eqs. 35 and 34). The per capita amount of human capital dedicated to the production of goods does not show a clear pattern because the increase of per capita human capital tend to raise the human capital dedicated to the production of goods but the rise of the tax rate have the opposite effect because it increases the demand of teachers and bureaucrats (see eq. 33). These results are aligned with those in the benchmark model. It follows from equations (33), (35) and (34) that the shares of human capital dedicated to production; bureaucrats; and

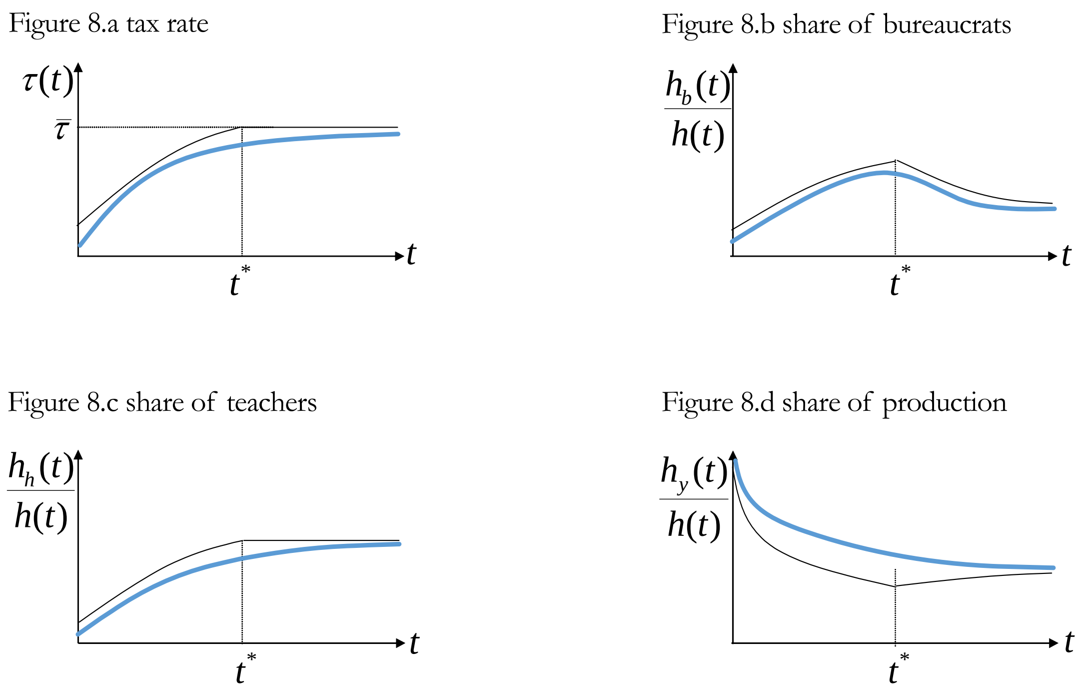

Fig. 8.— Equilibrium (thin line) versus optimal (thick line) allocation of human capital

teachers are as follows:

$$\frac{h_y}{h} = (1-\tau) \tag{41}$$

$$\frac{h_b}{h} = \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \frac{1}{h} \tag{42}$$

$$\frac{h_h}{h} = \tau - \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \frac{1}{h} \tag{43}$$

From the above equations we observe that the share of teachers over total human capital, h_h/h, increases along the transition; the fraction of human capital dedicated to production over total human capital, h_y/h, decreases; whereas the behavior of the fraction of bureaucrats in total human capital, h_b/h, is ambiguous (see Figure 8). Thus, these results show that the fact that in the first stage of development an increasing part of human capital is devoted to public sector activities, such as bureaucracy and public education, instead private sector is not a sign of bad allocation of resources; on the contrary, it is consistent with the efficient allocation.

9. Implications of the model and empirical evidence

Here we discuss the main implication of the proposed theory in the light of the literature and the empirical evidence available. These are mainly two: the tight link between public education and bureaucracy, and the relationship between the production of final goods and human capital.

9.1. Bureaucracy and public education

This model proposes a theory of reallocation of human capital such that at the beginning of the development the government absorbs larger portion of human capital compared with later stages. The reason for this is that the government requires human capital to constitute the bureaucracy, which allows it to collect taxes to finance public transfers and public education. Thus, the government hires teachers to educate workers, and a share of these skilled workers will end up working for the government as teachers and as bureaucrats. The remaining amount of skilled workers are devoted to the private sector producing goods. Therefore, our theory suggests that the public education system is developed in societies at the same time as bureaucracy because the government requires educated workers to carry out its activity.

This vision is shared by the mainstream literature of political science and the Weberian "ideal type" of bureaucracy. Studies on the rise and evolution of the bureaucracy sector point out that governments demand competent administration, both to provide sufficient public goods to population to avoid removal from power (Przeworski, Stokes and Manin, 1999; and Bueno de Mesquita et al., 2003) and/or to extract as much value from the population as possible (McGuire and Olson, 1996). More recently, as we have already commented in the introduction, Hollyer (2011) finds evidence that the change of the recruitment process in the bureaucratic sector in Western Europe from patronage to meritocratic is strongly and positively related to the widespread development of the education system.

To illustrate the relationship between the size of bureaucracy, educational sector and per capita GDP, we have used the more recent data of the International Labor Office (ILO). We have selected a wide sample of developing and developed economies in 2009. ILO uses definitions of the International Standard Industrial Classification of All Economic Activities (ISIC Rev. 3) from UN. Hence, the size of bureaucracy is measured as the percentage of public officers working in defense; compulsory social security; and public administration over total employment. The size of the educational sector is measured as the percentage of teachers and professors over total employment. Figure 9 shows the results.

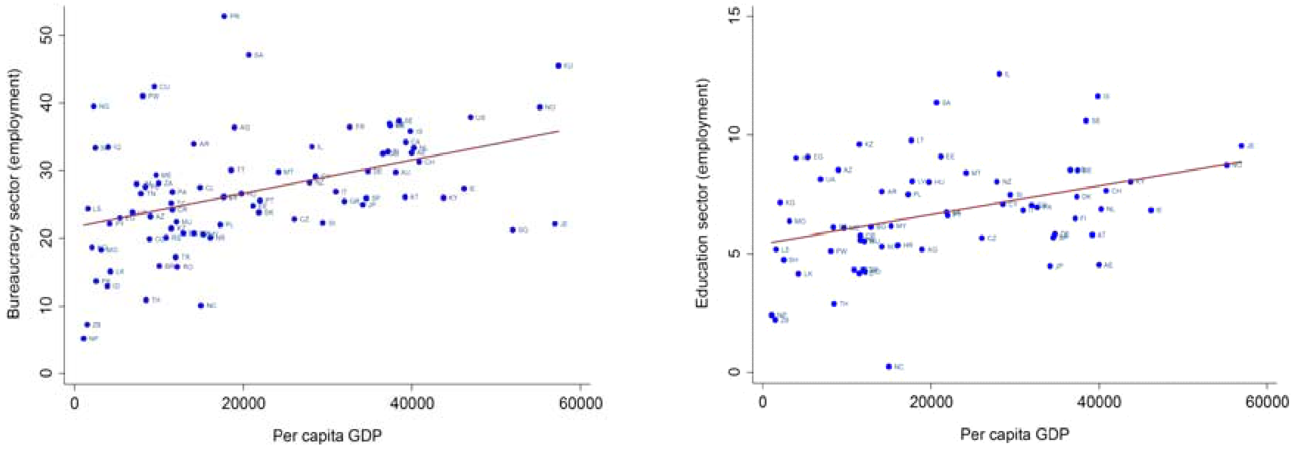

Fig. 9.— Bureaucracy and education in developed and developing countries

A clear and positive correlation emerges between both the size of bureaucracy and educational sector with per capita GDP. From a neoclassical point of view, we could assert that the development process is characterized by a rise in the amount of human capital allocated to education and bureaucracy.

9.2. The size and the effectiveness of the bureaucratic sector

Our theory shows that an improvement in the technology of the bureaucracy (institutional improvement) produces a reallocation of human capital from bureaucracy to the private sector, which produces a reduction in the wage of skilled workers and discourages the accumulation of human capital. Finally, the economy reaches a higher level of GDP but a lower level of human capital. This implies a negative relationship between per capita human capital and institutional quality, and a positive relationship between institutional quality and per capita GDP. As we explained in Section 7, these relationships together may generate a misleading negative relationship between per capita human capital and per capita GDP: countries with better institutions show higher levels of per capita production and GDP but lower levels of human capital than those with poorer institutions.

To test this finding, we have estimated the impact of the bureaucracy effectiveness in determining the size of bureaucracy for a wide sample of developing and developed countries. Given that there exists a positive relationship between human capital level and the size of bureaucracy (see Figure 9), we need control for this effect. Thus, we estimate the following equation using ordinary least square:

$$Bureau = \beta_0 + \beta_1\ Quality + \beta_2\ pcGDP$$

The size of the bureaucracy (*Bureau*) is obtained from the ILO dataset (see Subsection 9.1 for details) in year 2009, which is measured as the share of public sector employment over total employment. We use two different measures of public sector employment provided by ILO: the first , *Bureau1*, covers employment in the government sector plus employment in publicly-owned resident enterprises and companies, operating at local; state (or regional); and central levels of government. It covers all individuals employed directly by those institutions, regardless of the particular type of employment contract. The second, *Bureau2*, covers all employment of general government sector as defined in System of National Accounts 1993 plus employment of publicly owned enterprises and companies, resident and operating at local; state (or regional); and central levels of government. It also covers all individuals employed directly by those institutions, without regard for the particular type of employment contract.

Human Capital level is obtained from the Penn World Table 9.0 database in year 2009. It is approximated as the human capital index (in logs), based on returns to education and years of schooling (see human capital in PWT 9.0 for details).

Finally, the quality of bureaucracy (*Quality*) is proxied by three different measures: "Government effectiveness"; "Rule of law"and "; and Control of corruption", all of which are provided by the Worldwide Governance Indicators (World Bank). "Government effectiveness"(*GE*) takes into account perceptions of the quality of policy formulation and implementation; the quality of the civil service; the degree of its independence from political pressures and the quality of public services; and the credibility of the government's commitment to such policies. "Rule of law"(*ROL*) takes into account perceptions of the extent to which individuals have confidence by the rules of society, and in particular the quality of the police and the courts; property rights and contract enforcement; and the likelihood of violence and crime. "Control of corruption"(*CC*) considers perceptions of the extent to which public power is exercised for private gain, including both grand and petty forms of corruption, and capture of the state by private interest and elites. These three measures range from -2.5 (weak) to 2.5 (strong) governance performance. We also take the observations of these variables in 2009.

Table 1 shows the results of the estimation when we use *Bureau1* to measure the size of the bureaucracy. The sample size is of about 40 countries (developed and developing ones).

Our results confirm the predictions of the model. We clearly observe that there is a positive and statistically significant relationship between the human capital level in the

Table 1: Bureaucracy size is measured by *Bureau1*.

	Coef.	Std.Err.	t	$p > \|t\|$
GE	-1.08	1.41	-0.76	0.451
HK	31.78	7.10	4.48	0.000
$cons$	-14.60	7.32	-1.99	0.054
ROL	-0.26	1.31	-0.20	0.843
HK	29.72	7.38	4.03	0.000
$cons$	-12.72	7.73	-1.65	0.108
CC	-1.29	1.23	-1.05	0.300
HK	32.35	6.80	4.76	0.000
$cons$	-15.40	7.17	-2.15	0.038

Countries in the sample: Argentina; Armenia; Belgium; Bulgaria; Canada; Colombia; Chile; Dominican Republic; El Salvador; Egypt; Estonia; France; Greece; Hungary; Japan; Kazakhstan; Lithuania; Mexico; Malaysia; Moldova (Republic of); Mongolia; Morocco; Norway; Panama; Paraguay; Poland; Philippines; Peru; Romania; Serbia; Slovenia; Slovakia; South Africa; Spain; Sri Lanka; Ukraine; United Kingdom; Uruguay; Venezuela (Bolivarian Republic of).

economy and the size of the bureaucracy. There is also a negative relation between the institutional quality and the the size of the bureaucracy. Although the level of significance is low, we observe that the negative sign of the relationship is robust to the three measures we use (government effectiveness, rule of law and control of corruption).

Table 2 displays the results of the estimation when we use *Bureau2* to proxy the size of the bureaucracy. In this case, the sample size is of about 38 countries. We observe the same patterns as those displayed at Table 1: a positive (negative) relationship between the size of the bureaucracy and the human capital (institutional quality) level. Again, the significance of the institutional quality seems to be small; however, the negative sign is the correct one for the three alternative measures that we use.

9.3. Bureaucracy and human capital relationship

Our model predicts a hump-shaped pattern in the share of human capital allocated in the public sector (bureaucracy) insofar as countries are accumulating human capital: in the first stage of development, when human capital rises, it becomes less scarce and this makes

Table 2: Bureaucracy size is measured by *Bureau2*.

	Coef.	Std.Err.	t	$p > \|t\|$
GE	-1.36	1.49	-0.91	0.368
HK	19.43	9.26	2.10	0.043
$cons$	-0.13	9.92	0.01	0.989
ROL	-0.41	1.35	-0.31	0.761
HK	16.70	9.24	1.81	0.079
$cons$	2.44	9.99	0.24	0.808
CC	-1.17	1.15	-1.01	0.318
HK	19.10	8.91	2.14	0.039
$cons$	0.20	9.74	0.02	0.984

Countries in the sample: Armenia; Australia; Botswana; Bulgaria; Brazil; Czech Republic; Cyprus; Croatia; Costa Rica; Canada; Denmark; Estonia; Egypt; Greece; Germany; Hong Kong, China; Ireland; Japan; Kyrgyz Republic; Latvia; Lithuania; Luxembourg; Macau, China; Malta; Mexico; Moldova (Republic of); New Zealand; Paraguay; Poland; Russian Federation; Serbia; Slovakia; Sweden; Spain; Turkey; Ukraine; United Kingdom; United States.

the government to hire more bureaucrats who then implement a higher effective tax rate. Once that the statutory tax rate is reached, the tax rate is fixed independently of per capita human capital (see Figure 1.a). Thus, the government only hires the amount of bureaucrats needed to collect the statutory tax rate. This implies that the share of bureaucrats in human capital decreases when per capita human capital rises.

To test this finding, we have estimated the impact of the human capital accumulation in determining the fraction skilled workers that are allocated in the of bureaucracy sector for a wide sample of developing and developed countries. The following equation is estimated using ordinary least square:

$$BureauShare = \beta_0 + \beta_1 \ HumanCapitalShare + \beta_2 \ HumanCapitalShare^2$$

The share of skilled workers allocated in the bureaucracy (*BureauShare*) is obtained from the ILO data set (see Subsection 9.1 for details) in year 2009. It is measured as the share of public sector employment over total amount of skilled workers. We use two different measures of public sector employment provided by ILO defined in previous section: *Bureau1* and *Bureau2*. The amount of skilled workers is obtained from the ILO dataset (see Subsection 9.1 for details) in year 2009, which is defined as the total amount of employed workers with advanced education and high school education. Thus, the human capital share in the economy (*HumanCapitalShare*) is obtained as the fraction of skilled workers over total employed workers.

Table 3 shows the results of the estimation when we use *Bureau1* and *Bureau2*. The sample size is of about 51 and 38 countries (developed and developing ones) respectively.

Table 3: Bureaucracy and Human Capital Share.

	Coef.	Std.Err.	t	$p > \|t\|$
		Bureau1		
HKS	0.41	0.53	0.77	0.448
HKS^2	-0.004	0.004	-1.07	0.291
cons	23.73	16.07	1.48	0.149
		Bureau2		
HKS	0.54	0.56	0.96	0.334
HKS^2	-0.005	0.004	-1.27	0.214
cons	20.22	16.95	1.19	0.241

These results confirm predictions of the model. We observe that exists a hump-shaped form between the share of human capital allocated in the bureaucracy sector and the human capital level in the economy: the linear term has a positive effect whereas the square term has a negative sign. Though levels of significance are low, we observe that the sign of the relationship is the same using the two different measures of bureaucracy.

According to our model, countries with better institutions show larger levels of per capita GDP but smaller levels of human capital and bureaucracy than those with poorer institutions (see the previous subsection). We would like to prove the robustness of the hump-shaped relationship that we have obtained to the institutional quality effect. We now estimate the following equation using ordinary least square:

$$BureauShare = \beta_0 + \beta_1\ HumanCapitalShare + \beta_2\ HumanCapitalShare^2 + \beta_3\ Quality$$

Quality of bureaucracy (*Quality*) is proxied by the three different measures described above: "Government effectiveness"; "Rule of law"; and "Control of corruption".

Table 4 shows the results of the estimation when we use *Bureau1* to measure the size of the bureaucracy and Table 5 when we use *Bureau2*.

Table 4: Bureaucracy size is measured by *Bureau1*.

	Coef.	Std.Err.	t	$p > \|t\|$
GE	-0.86	2.47	-0.35	0.730
HK	0.42	0.55	0.77	0.447
HKS^2	-0.004	0.004	-1.02	0.314
$cons$	22.55	16.61	1.36	0.184
ROL	-0.11	2.19	-0.05	0.960
HK	0.41	0.56	0.73	0.469
HKS^2	-0.004	0.004	-1.02	0.315
$cons$	23.21	14.24	1.35	0.188
CC	-1.44	2.03	-0.71	0.482
HK	0.43	0.54	0.79	0.434
HKS^2	-0.004	0.004	-1.02	0.314
$cons$	21.83	16.46	1.33	0.194

Again, we observe that there is a hump-shaped relation between the share of the bureaucracy and the human capital level in the economy. There is also a negative relationship between the fraction of the bureaucracy and the institutional quality. Although the level of significance is low, we observe that signs of the relationships are robust to the three measures we use (government effectiveness, rule of law and control of corruption).

10. Conclusion

The role of human capital in development and economic growth is surprisingly controversial. While at the micro level the empirical literature reports significant returns to increases in education consistent with the theory, macro analysis finds not only a weak relation between economic performance and human capital but some studies also show a negative

Table 5: Bureaucracy size is measured by *Bureau2*.

	Coef.	Std.Err.	t	$p > \|t\|$
GE	-1.20	2.60	-0.46	0.647
HK	0.56	0.58	0.97	0.340
HKS^2	-0.005	0.004	-1.21	0.235
$cons$	18.63	17.48	1.07	0.294
ROL	-0.30	2.31	-0.13	0.896
HK	0.55	0.59	0.93	0.358
HKS^2	-0.005	0.004	-1.22	0.232
$cons$	19.20	18.17	1.06	0.298
CC	-1.55	2.14	-0.72	0.474
HK	0.56	0.57	0.98	0.333
HKS^2	-0.005	0.004	-1.21	0.234
$cons$	18.16	17.35	1.05	0.303

impact of human capital.

In this article, we offer a novel explanation to understand why the weak relationship between economic growth and human capital. This paper builds a theory which explains how human capital is allocated in the economy during the development process. This allow us to understand how the allocation of human capital has evolved among different activities, and among public and private sectors when economies are growing. We build a model in which the public educational system is a key factor affecting the accumulation of human capital. To finance public education, the government needs to hire skilled workers to work as bureaucrats and collect taxes. Thus, the government absorbs part of the human capital of the economy: it hires bureaucrats to collect taxes and it also hires teachers for the public education system. When per capita human capital is low, the scarcity of human capital prevents the government to hire enough bureaucrats to implement the statutory tax rate. The resulting low effective tax rate implies that it is not possible to hire many teachers and, consequently, most human capital is used in the production of goods (in the private sector). As the transition proceeds, human capital rises, which makes human capital more abundant and allows the government to hire more bureaucrats to implement a higher effective tax rate and to hire more teachers for the public education system. The fact that an increasing part of human capital is recruited by the government during the development process, diverting human capital from the private sector, may involve a slowdown of production in the private sector. However, this does not necessarily mean that this absorption of human capital by the government is wasteful or inefficient. On the contrary, we analyzed the human capital optimal allocation and we show that the efficient allocation follows the same pattern as in the benchmark model: an increasing part of human capital is absorbed by the public sector at the expense of the private sector, implying an increasing optimal tax rate along the transition.

This paper emphasizes the essential role that differences in institutions have in understanding the weak effect of human capital on macroeconomic performance documented in the literature. Differences in institutions across countries may lead to a spurious empirical relationship between human capital and economic performance at the macro level. Countries with poor institutions require more bureaucrats. This increases the incentive to expand the education system and it also increases the amount of human capital, but it reduces the steady state levels of production of goods and GDP. Thus, countries with weaker institutions would have more human capital and less GDP than countries with stronger institutions. Consequently, a spurious negative correlation between GDP and human capital may arise among countries with different levels of institutional quality.

The implications and results of our model are consistent with the available empirical evidence. Our theory shows that the bureaucracy system increases at the same time that the education system expands, and it demands a large amount of human capital in that early stage of development. Hollyer (2011) shows strong evidence on this regard. He finds that bureaucracy disproportionately expands in countries (measured thought the change of recruitment system from exclusive patronage to merit-based system) when education is widespread among population. Thus, in a first stage of development, bureaucracy increases dramatically simultaneously to the development of educational programs. Second, for a wide sample of developing and developed countries, we prove that a spurious relationship between human capital and GDP may arise from the data. More precisely, our empirical analysis evidences that the size of the bureaucracy is positively related to the per capita GDP level whereas it is negatively related to the institutional quality of the government. This result sheds light on the apparent weak role of human capital on economic performance. Finally, our estimations seem to support the model's prediction of a hump-shaped relationship between the fraction of human capital devoted to the bureaucracy and the human capital level in the economy. Our model suggest that the observed high fraction of human capital allocated in the public sector in the first stage of development is a consequence of the government's efforts to increase the effective tax rate.

11. References

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12. Appendix

Dynamic behavior: Dynamic system of the economy

The Hamiltonian associated with the optimization problem (4) may be written as follows:

$$u\left(w_h(t)\left(1-\tau(t)\right)h(t)+w(t)\left(1-h(t)-s(t)\right)+tr(t)\right)e^{-(\rho-n)t}+\lambda(t)e^{-(\rho-n)t}\left[\mu(t)s\left(t\right)-bh\left(t\right)\right]$$

The FOCs are as follows:

$$c(t)^{-\sigma}w(t) = \lambda(t)\mu(t) \Rightarrow \lambda(t) = \frac{c(t)^{-\sigma}w(t)}{\mu(t)} = c(t)^{-\sigma}p_h(t) \tag{44}$$

$$\dot{\lambda}(t)-(\rho-n)\lambda(t) = -c(t)^{-\sigma}\left[w_h(t)\left(1-\tau(t)\right)-w(t)\right]+\lambda b \tag{45}$$

Note that we have not derived with respect to $\mu(t)$ because this is an "aggregate" variable that does not depend on the decision of an individual household. Using these equations, we obtain:

$$\begin{aligned}\frac{\dot{\lambda}(t)}{\lambda(t)} &= -\sigma\frac{\dot{c}(t)}{c(t)}+\frac{\dot{p}_h(t)}{p_h(t)} \Rightarrow \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}\left[\frac{w_h(t)\left(1-\tau(t)\right)-w(t)}{p_h(t)}+\frac{\dot{p}_h(t)}{p_h(t)}-(\rho+m)\right]\end{aligned}$$

where we have used the fact that $n=b-m$. Using eqs. (10) and (11), the goods market clear condition, and the definition of $\mu(t)$ it follows that:

$$\begin{aligned}&\left[\frac{\xi}{s(t)}+\frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)}\right]\dot{s}(t) = \\ &-\left(\frac{h_h(t)}{s(t)}\right)^{\xi}\left[\frac{\alpha\left(1-\tau(t)\right)\left(1-h(t)-s(t)\right)-(1-\alpha)\,h_y(t)}{(1-\alpha)\,h_y(t)}\right]+(\rho+m)+ \\ &+\left[\frac{\xi}{h_h(t)}\frac{\partial h_h(h(t))}{\partial h}-\frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)}+\frac{\alpha(\sigma-1)}{h_y(t)}\frac{\partial h_y(h(t))}{\partial h}\right]\dot{h}(t)\end{aligned}$$

Thus, the dynamics of the economy are described by the following the dynamic system:

$$\left[\frac{\xi}{s(t)} + \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)}\right]\dot{s}(t) = \tag{46}$$
$$-\left(\frac{h_h(t)}{s(t)}\right)^{\xi}\left[\frac{\alpha\,(1-\tau(t))\,(1-h(t)-s(t))-(1-\alpha)\,h_y(t)}{(1-\alpha)\,h_y(t)}\right] + (\rho+m) +$$
$$+\left[\frac{\xi}{h_h(t)}\frac{\partial h_h(h(t))}{\partial h} - \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)} + \frac{\alpha(\sigma-1)}{h_y(t)}\frac{\partial h_y(h(t))}{\partial h}\right]\underbrace{\left[(h_h(h(t))^{\xi}\,s(t)^{1-\xi}-bh(t)\right]}_{\dot{h}(t)}$$
$$\dot{h}(t) = (h_h(h(t))^{\xi}\,s(t)^{1-\xi} - bh(t) \tag{47}$$

Proof of Proposition 3

The dynamic system (46)-(47) implies that the following two equations should hold at the steady state:

$$\left.\begin{array}{l}\left(\frac{h_h(h)}{s}\right)^{\xi}\left[\frac{\alpha(1-\tau)(1-h-s)-(1-\alpha)h_y}{(1-\alpha)h_y}\right] = (\rho+m)\\ h_h(h)^{\xi}s^{1-\xi} = bh\end{array}\right\} \Rightarrow \Psi(h) = 0$$

where $\Psi(h)$ is the function that defines the steady state:

$$\Psi(h) = \left(\frac{h_h(h)}{bh}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\,(1-\tau(h))\left(1-h\left(1+b\left(\frac{bh}{h_h(h)}\right)^{\frac{\xi}{1-\xi}}\right)\right)-(1-\alpha)\,h_y}{(1-\alpha)\,h_y}\right] - (\rho+m)$$

Function $\Psi(h)$ has two branches.

If $h \leq \overline{h}$:

$$\Psi(h) = \left(\frac{\lambda(\gamma^\gamma \Gamma)^{\frac{\gamma}{1-\gamma}}}{b}\right)^{\frac{\xi}{1-\xi}} h^{\frac{\gamma\xi}{(1-\gamma)(1-\xi)}} \times \left[\frac{\alpha\left(1-\left(\gamma\Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}\right)\left(1-h\left(1+b\left(\frac{bh}{\lambda(\gamma^\gamma\Gamma h)^{\frac{1}{1-\gamma}}}\right)^{\frac{\xi}{1-\xi}} \frac{1}{h^{\frac{\gamma\xi}{(1-\gamma)(1-\xi)}}}\right)\right)-(1-\alpha)\left[h-(\gamma+\lambda)(\gamma^\gamma\Gamma h)^{\frac{1}{1-\gamma}}\right]}{(1-\alpha)\left(h-(\gamma+\lambda)(\gamma^\gamma\Gamma h)^{\frac{1}{1-\gamma}}\right)}\right] -(\rho+m) =$$

$$\frac{\left(\frac{\lambda(\gamma^\gamma\Gamma)^{\frac{\gamma}{1-\gamma}}}{b}\right)^{\frac{\xi}{1-\xi}}}{h^{\frac{1-\gamma-\xi}{(1-\gamma)(1-\xi)}}} \left[\frac{\alpha\left(1-(\gamma^\gamma\Gamma)^{\frac{1}{1-\gamma}} h^{\frac{\gamma}{1-\gamma}}\right)}{(1-\alpha)\left(1-(\gamma+\lambda)(\gamma^\gamma\Gamma)^{\frac{1}{1-\gamma}} h^{\frac{\gamma}{1-\gamma}}\right)} \left(1-h-b\left(\frac{b}{\lambda(\gamma^\gamma\Gamma)^{\frac{1}{1-\gamma}}}\right)^{\frac{\xi}{1-\xi}} h^{\frac{1-\gamma-\xi}{(1-\gamma)(1-\xi)}}\right)-h\right] -(\rho+m) \quad (48)$$

If $h \geq \overline{h}$:

$$\Psi(h) = \left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}} \left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)} \frac{1-h\left(1+\left(\frac{b}{(\lambda\overline{\tau})^\xi}\right)^{\frac{1}{1-\xi}}\right)}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right] - (\rho+m) \quad (49)$$

It follows from eq. (48) that if $\gamma+\xi \leq 1$, then $\Psi(h)$ is strictly decreasing. Consequently, if there is steady state, then it is unique. To have a steady state h^{ss} such that $h^{ss} > \overline{h}$, the following condition should hold:

$$\Psi(\overline{h}) = \left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}} \left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)} \frac{1-\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^\xi}\right)^{\frac{1}{1-\xi}}\right)}{\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}(1-\overline{\tau}(\lambda+\gamma))} - 1\right] - (\rho+m) > 0 \Leftrightarrow$$

$$\Gamma > \overline{\Gamma} \equiv \left[\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}\left[\left(1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right)(1-\overline{\tau}(\lambda+\gamma))+\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^\xi}\right)^{\frac{1}{1-\xi}}\right)\right]}{\gamma}\right]^\gamma$$

Finally, note that for h close enough to $\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^\xi}\right)^{\frac{1}{1-\xi}}\right)^{-1}$, the function $\Psi(h)$ becomes negative. Thus, if $\Gamma > \overline{\Gamma}$ we can guarantee that there is a unique steady state and $h^{ss} > \overline{h}$ at such steady state. ■

Proof of Proposition 4

The dynamic system (46)-(47) when $h(t) > \overline{h}$ is as follows:

$$\dot{h}(t) = F_h(h(t), s(t)) \tag{50}$$
$$\dot{s}(t) = F_s(h(t), s(t)) \tag{51}$$

where:

$$F_h(h,s) = (\lambda\overline{\tau}h)^{\xi} s^{1-\xi} - bh$$

$$F_s(h,s)\left[\frac{\xi}{s} + \frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] = -\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)h - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right] + (\rho+m) +$$

$$+\left[\frac{\xi}{h} + \frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\left[\frac{\alpha(\sigma-1)(1-\overline{\tau}\lambda)}{1+(\sigma-1)(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)h - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right]\right] F_h(h,s)$$

First, we will prove that the locus $\dot{s}(t) = F_s(h(t), s(t)) = 0$ exists. Note that:

$$\frac{\partial F_s(h^{ss}, s^{ss})}{\partial s} =$$

$$\left[\frac{\xi}{s^{ss}} + \frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right]^{-1} \times \left\{(\rho+m)\left[\frac{\xi}{s^{ss}} + \frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1}{(1-\overline{\tau}\lambda)h^{ss} - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h^{ss} - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1}\right] + \right.$$

$$\left. + \left[\frac{\xi}{h^{ss}} + \frac{1+(\sigma\text{-}1)(1\text{-}\alpha)}{1-h^{ss}-s^{ss}}\left[\frac{\alpha(\sigma\text{-}1)(1\text{-}\overline{\tau}\lambda)}{1+(\sigma\text{-}1)(1\text{-}\alpha)}\frac{1-h^{ss}-s^{ss}}{(1\text{-}\overline{\tau}\lambda)h^{ss}\text{-}\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right]\right](1\text{-}\xi)\left(\frac{\lambda\overline{\tau}h^{ss}}{s^{ss}}\right)^{\xi}\right\} > 0$$

Thus, it follows from the Implicit Function Theorem that in a surrounding of the steady state it is possible to define $s^{\dot{s}=0}(h) \Leftrightarrow F_s\left(h, s^{\dot{s}=0}(h)\right) = 0$.

Secondly, we will prove that the locus $\dot{s}(t) = F_s(h(t), s(t)) = 0$ is above the locus $\dot{h}(t) = F_h(h(t), s(t)) = 0$ when $h < h^{ss}$. Let us define $s^{\dot{h}=0}(h) \Leftrightarrow F_h\left(h, s^{\dot{h}=0}(h)\right) = 0 \Leftrightarrow s^{\dot{h}=0}(h) = \left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}} h$. Note that $F_s\left(h, s^{\dot{h}=0}(h)\right)\left[\frac{\xi}{s^{\dot{h}=0}(h)} + \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s^{\dot{h}=0}(h)}\right] = -\Psi(h)$ (see eq. 49). Thus, it follows from the proof of proposition **3** that

$$\left\{\begin{array}{l} \text{If } h < h^{ss} \Rightarrow F_s\left(h, s^{\dot{h}=0}(h)\right) < 0 \\ \text{If } h > h^{ss} \Rightarrow F_s\left(h, s^{\dot{h}=0}(h)\right) > 0 \end{array}\right\} \tag{52}$$

Now, we need to prove that in a surrounding of the steady state, when $h < h^{ss}$, for some $s(t) > s^{\dot{h}=0}(h)$ it is possible to obtain that $F_s(h(t), s(t)) > 0$. Note that:

$$\frac{F_s(h,s)}{F_h(h,s)}\left[\frac{\xi}{s}+\frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] =$$

$$\frac{1}{F_h(h,s)}\left\{-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+(\rho+m)\right\}+ \tag{53}$$

$$+\left[\frac{\xi}{h}+\frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\left[\frac{\alpha(\sigma-1)(1-\overline{\tau}\lambda)}{1+(\sigma-1)(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]\right] \tag{54}$$

We know that if $\frac{F_s(h,s^{ss})}{F_h(h,s^{ss})} > 0$ when $h < h^{ss}$ then $F_s(h(t), s(t)) > 0$ when $s(t) = s^{ss}$ which, in turn, implies that there is $s^{\dot{s}=0}(h) \in \left(s^{\dot{h}=0}(h), s^{ss}\right)$. Note that the first term of eq. (54) is always positive:

$$\lim_{h\to h^{ss}} \frac{-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+(\rho+m)}{F_h\left(h,s^{ss}\right)} =$$

$$\lim_{h\to h^{ss}} \frac{\frac{\partial\left[-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+(\rho+m)\right]}{\partial h}}{\frac{\partial F_h(h,s^{ss})}{\partial h}}$$

$$\frac{(\rho+m)\left[-\frac{\xi}{h^{ss}}+\frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\left[\frac{1}{1-h^{ss}-s^{ss}}+\frac{(1-\overline{\tau}\lambda)}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\right]}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1}\right]}{-(1-\xi)b} =$$

$$\frac{(\rho+m)\left[\frac{\left[1+\frac{(\rho+m)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}\right]\left[\frac{1}{1-h^{ss}-s^{ss}}+\frac{(1-\overline{\tau}\lambda)}{(1-\overline{\tau}\lambda)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\right]}{\left[\frac{(\rho+m)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}\right]}-\frac{\xi}{h^{ss}}\right]}{(1-\xi)b} >$$

$$\frac{\left[\frac{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}+(\rho+m)}{1-h^{ss}-s^{ss}}+\frac{(\rho+m)(1-\xi)+\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}{h^{ss}}\right]}{(1-\xi)b} > 0$$

Now we need to prove that the second term of eq. (54) is also positive. Let us define $\overline{\sigma}$ such that:

$$\overline{\sigma} \overset{Def}{\Leftrightarrow} \lim_{h\to h^{ss}} \frac{F_s\left(h,s^{ss}\right)}{F_h\left(h,s^{ss}\right)} \left[\frac{\xi}{s^{ss}} + \frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right] = 0 \Leftrightarrow$$

$$\frac{\xi}{h^{ss}} + \frac{1+(\overline{\sigma}-1)(1-\alpha)}{1-h^{ss}-s^{ss}} \left[\frac{\alpha(\overline{\sigma}-1)\left(1-\overline{\tau}\lambda\right)}{1+(\overline{\sigma}-1)(1-\alpha)} \frac{1-h^{ss}-s^{ss}}{\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right] =$$

$$-\lim_{h\to h^{ss}} \frac{-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right] + (\rho+m)}{F_h\left(h,s^{ss}\right)} \equiv -\Omega \Leftrightarrow$$

$$\left[\alpha(\sigma-1)\left(1-\overline{\tau}\lambda\right)\frac{1-h^{ss}-s^{ss}}{\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - \left[1+(\sigma-1)(1-\alpha)\right]\right] = -\left[\Omega+\frac{\xi}{h^{ss}}\right]\left(1-h^{ss}-s^{ss}\right)$$

$$\sigma\left[\alpha\left(1-\overline{\tau}\lambda\right)\frac{1-h^{ss}-s^{ss}}{\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - (1-\alpha)\right] =$$

$$-\left[\Omega+\frac{\xi}{h^{ss}}\right]\left(1-h^{ss}-s^{ss}\right) + \alpha\left[\left(1-\overline{\tau}\lambda\right)\frac{1-h^{ss}-s^{ss}}{\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} + 1\right]$$

Therefore

$$\overline{\sigma} = 1 + \frac{1-\left[\Omega+\frac{\xi}{h^{ss}}\right]\left(1-h^{ss}-s^{ss}\right)}{\alpha\left(1-\overline{\tau}\lambda\right)\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - (1-\alpha)}$$

where $\alpha\left(1-\overline{\tau}\lambda\right)\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - (1-\alpha) > \alpha\left(1-\overline{\tau}\lambda\right)\frac{(1-\alpha)}{\alpha(1-\overline{\tau})} - (1-\alpha) = (1-\alpha)\left[\frac{\overline{\tau}(1-\lambda)}{(1-\overline{\tau})}\right] > 0.$

Thus, if $\sigma > \overline{\sigma}$ then in a surrounding of h^{ss}, $(h^{ss}-\varepsilon, h^{ss}+\varepsilon)$, such that:

$$\text{If } h \neq h^{ss} \text{ then } \frac{F_s\left(h,s^{ss}\right)}{F_h\left(h,s^{ss}\right)}\left[\frac{\xi}{s} + \frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] > 0 \Rightarrow$$

$$\left\{\begin{array}{l} \text{If } h < h^{ss} \Rightarrow F_h\left(h,s^{ss}\right) > 0 \Rightarrow F_s\left(h,s^{ss}\right) > 0 \\ \text{If } h > h^{ss} \Rightarrow F_h\left(h,s^{ss}\right) < 0 \Rightarrow F_s\left(h,s^{ss}\right) < 0 \end{array}\right\} \tag{55}$$

Eqs. (52) and (55) imply that:

$$\left\{\begin{array}{l} \text{If } h < h^{ss} \Rightarrow s^{\dot{s}=0}(h) \in \left(s^{\dot{h}=0}(h), s^{ss}\right) \\ \text{If } h > h^{ss} \Rightarrow s^{\dot{s}=0}(h) \in \left(s^{ss}, s^{\dot{h}=0}(h)\right) \end{array}\right\} \tag{56}$$

This implies that $s^{\dot{h}=0}(h)$ is below $s^{\dot{s}=0}(h)$ when $h(t) < h^{ss}$:

$$\frac{\partial s^{\dot{s}=0}(h)}{\partial h} = -\frac{\frac{\partial F_s(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_s(h^{ss},s^{ss})}{\partial s}} < \frac{\partial s^{\dot{h}=0}(h)}{\partial h} = -\frac{\frac{\partial F_h(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_s(h^{ss},s^{ss})}{\partial s}} \tag{57}$$

The dynamic system in a surrounding of the steady state may be linearized as follows:

$$\begin{bmatrix} \dot{h}(t) \\ \dot{s}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F_h(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_h(h^{ss},s^{ss})}{\partial s} \\ \frac{\partial F_s(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_s(h^{ss},s^{ss})}{\partial s} \end{bmatrix} \begin{bmatrix} h(t) - h^{ss} \\ s(t) - s^{ss} \end{bmatrix}$$

Eigenvalues are defined as follows:

$$\begin{vmatrix} \frac{\partial F_h(h^{ss},s^{ss})}{\partial h} - \lambda & \frac{\partial F_h(h^{ss},s^{ss})}{\partial s} \\ \frac{\partial F_s(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_s(h^{ss},s^{ss})}{\partial s} - \lambda \end{vmatrix} = \lambda^2 - tr\,\lambda + Det = 0 \Rightarrow \lambda = \frac{tr \pm \sqrt{tr^2 - 4Det}}{2}$$

It follows from (57) that:

$$\begin{aligned}
tr &= \frac{\partial F_h(h^{ss}, s^{ss})}{\partial h} + \frac{\partial F_s(h^{ss}, s^{ss})}{\partial s} \\
Det &= \underbrace{\frac{\partial F_h(h^{ss}, s^{ss})}{\partial h}}_{-} \underbrace{\frac{\partial F_s(h^{ss}, s^{ss})}{\partial s}}_{+} - \underbrace{\frac{\partial F_h(h^{ss}, s^{ss})}{\partial s}}_{+} \frac{\partial F_s(h^{ss}, s^{ss})}{\partial h} = \\
Det &= \frac{\partial F_s(h^{ss}, s^{ss})}{\partial s} \frac{\partial F_h(h^{ss}, s^{ss})}{\partial s} \left[\frac{\frac{\partial F_h(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_h(h^{ss},s^{ss})}{\partial s}} - \frac{\frac{\partial F_s(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_s(h^{ss},s^{ss})}{\partial s}} \right] = \\
Det &= \underbrace{\frac{\partial F_s(h^{ss}, s^{ss})}{\partial s}}_{+} \underbrace{\frac{\partial F_h(h^{ss}, s^{ss})}{\partial s}}_{+} \underbrace{\left[\frac{\partial s^{\dot{s}=0}(h^{ss})}{\partial h} - \frac{\partial s^{\dot{h}=0}(h^{ss})}{\partial h} \right]}_{-} < 0
\end{aligned}$$

Thus, one of the eigenvalues is positive and the other is negative. This means that the steady state is a saddle point. Furthermore, it follows from (56) that $s(t)$ is increasing when $h(t) < h^{ss}$ and decreasing when $h(t) > h^{ss}$.

Proof proposition 5

From eq. (29) it is easy to see that the skill premium evolves according to the following equation:

$$\dot{sp}(t) = -\frac{\alpha}{1-\alpha} \frac{1 - h(t) - s(t)}{h_y(t)} \left[\frac{\dot{h}_y(t)}{h_y(t)} + \frac{\dot{h}(t) + \dot{s}(t)}{1 - h(t) - s(t)} \right] \tag{58}$$

If follows from eq. (22) that:

$$\dot{sp} = \begin{cases} -\frac{\alpha}{1-\alpha}\frac{1-h-s}{h_y}\left[\frac{\left[1\text{-}\frac{(\gamma+\lambda)}{1-\gamma}(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h(t)^{\frac{\gamma}{1-\gamma}}+\frac{h(t)\text{-}(\gamma+\lambda)(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h(t)^{\frac{1}{1-\gamma}}}{1-h-s}\right]\dot{h}}{h_y}+\frac{\dot{s}}{1-h-s}\right] & \text{if } h(t) < \overline{h} \\ -\frac{\alpha}{1-\alpha}\frac{1-h-s}{h_y}\left[\frac{(1\text{-}\overline{\tau}\lambda)}{h_y}+\frac{\dot{h}+\dot{s}}{1-h-s}\right] & \text{if } h(t) \geq \overline{h} \end{cases}$$

Note that if $h(t) < h^{ss}$ and $\dot{h_y} > 0$, then the skill premium is always decreasing (see eq. 58). This means that if either $h(t) \geq \overline{h} \equiv \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}$ or $1 \geq \frac{(\gamma+\lambda)}{1-\gamma}(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h(t)^{\frac{\gamma}{1-\gamma}} \Leftrightarrow h(t) \leq \left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}$ then h_y increases along the transition, and consequently the skill premium would be always decreasing. Thus, if the skill premium is not decreasing, then it would be when human capital is in the interval $\left(\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}, \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\right]$. Thus, the following condition is sufficient in order that the skill premium is always decreasing: either $\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}} < \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}} \Leftrightarrow \overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$ or

$$f(\tau) = \min_{h\in\left[\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}, \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\right]}\left[1-\frac{(\gamma+\lambda)}{1-\gamma}(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h^{\frac{\gamma}{1-\gamma}}+\frac{h\text{-}(\gamma+\lambda)(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h^{\frac{1}{1-\gamma}}}{1-h}\right] \geq 0 \tag{59}$$

The above function $f(\tau)$ is decreasing in τ. Let us define $\widehat{\tau}$ as follows:

$$\widehat{\tau} = \begin{cases} 1 & \text{if } f(1) \geq 0 \\ \max\{\widehat{\tau} \text{ such that } \forall\tau \leq \widehat{\tau},\ f(\tau) \geq 0\} & \text{if } f(1) < 0 \end{cases}$$

If $\tau \leq \widehat{\tau}$, the condition (59) holds. Furthermore, $f\left(\frac{1-\gamma}{\gamma+\lambda}\right) = \frac{\gamma}{\left(1-\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}\right)\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}} > 0$. Thus, $\widehat{\tau} > \frac{1-\gamma}{\gamma+\lambda}$.

Proof proposition 6

At the steady state (see eq. 49)

$$\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{1-\alpha}\left(\frac{l}{h_y}\right)-1\right] = m+\rho \Leftrightarrow \frac{l}{h_y} = \frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}}$$

$$\ln l-\ln h_y = \ln\left(\frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}}\right)$$

Thus:

$$\left.\begin{array}{l}\frac{\frac{\partial l}{\partial\Gamma}}{l}-\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y}=0\Rightarrow\frac{\frac{\partial l}{\partial\Gamma}}{l}=\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y}\\ \frac{\frac{\partial y}{\partial\Gamma}}{y}=\alpha\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y}+(1-\alpha)\frac{\frac{\partial l}{\partial\Gamma}}{l}\end{array}\right\}\Rightarrow\frac{\frac{\partial y}{\partial\Gamma}}{y}=\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y} \tag{60}$$

Note that at the steady state:

$$\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{1-\alpha}\left(\frac{1-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)h}{h_y}\right)-1\right] = m+\rho \Leftrightarrow$$

$$\frac{1}{h_y}-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\frac{h}{h_y} = \frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}} \Rightarrow$$

$$-\frac{1}{h_y}\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y}-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\frac{\partial\left(\frac{h}{h_y}\right)}{\partial\Gamma} = 0 \Leftrightarrow \frac{\partial\left(\frac{h}{h_y}\right)}{\partial\Gamma} = \frac{\frac{1}{h_y}\frac{\frac{\partial h_y}{\partial\Gamma}}{h_y}}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)} \tag{61}$$

The per capita *gdp* is as follows:

$$gdp = y\left[(1-\alpha)+\alpha\left(\frac{h}{h_y}\right)\right]\Leftrightarrow\ln gdp=\ln y+\ln\left[(1-\alpha)+\alpha\left(\frac{h}{h_y}\right)\right]$$

Using equations (60) and (61), it follows that at the steady state:

$$\frac{\frac{\partial gdp}{\partial \Gamma}}{gdp} = \frac{\frac{\partial y}{\partial \Gamma}}{y} - \frac{\alpha}{\left[(1-\alpha)+\alpha\left(\frac{h}{h_y}\right)\right]}\frac{\partial\left(\frac{h}{h_y}\right)}{\partial \Gamma}$$

$$\frac{\frac{\partial gdp}{\partial \Gamma}}{gdp} = \frac{\frac{\partial h_y}{\partial \Gamma}}{h_y} - \frac{\alpha}{\left[(1-\alpha)+\alpha\left(\frac{h}{h_y}\right)\right]}\frac{\frac{1}{h_y}\frac{\frac{\partial h_y}{\partial \Gamma}}{h_y}}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)} =$$

$$\left[1-\frac{\alpha}{[(1-\alpha)h_y+\alpha h]}\frac{1}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}\right]\frac{\frac{\partial h_y}{\partial \Gamma}}{h_y}$$

Thus, since $\frac{\partial h_y}{\partial \Gamma} > 0$:

$$\frac{\partial gdp}{\partial \Gamma} > 0 \Leftrightarrow \frac{\alpha}{[(1-\alpha)(1-\lambda\overline{\tau})+\alpha]h^{ss}-(1-\alpha)\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}}\frac{1}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)} < 1 \Leftrightarrow$$

$$\alpha < \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[h^{ss}-\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\lambda\overline{\tau}h^{ss}+\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}$$

Note that:

$$\frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[h^{ss}-\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\lambda\overline{\tau}h^{ss}+\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]} > \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\overline{h}-\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\lambda\overline{\tau}\overline{h}+\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]} =$$

$$\frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}-\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\lambda\overline{\tau}\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}+\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]} = \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[1-\gamma\overline{\tau}]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[\lambda+\gamma]\overline{\tau}}$$

Thus if:

$$\alpha < \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[1-\gamma\overline{\tau}]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[\lambda+\gamma]\overline{\tau}}$$

Then:

$$\alpha < \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[1-\gamma\overline{\tau}]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[\lambda+\gamma]\,\overline{\tau}} <$$

$$\frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[h^{ss}-\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\left[\lambda\overline{\tau}h^{ss}+\left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}\right]} \Rightarrow \frac{\partial gdp}{\partial\Gamma} > 0$$

Thus, there is $\widetilde{\alpha} \geq \frac{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[1-\gamma\overline{\tau}]}{1+\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)[\lambda+\gamma]\overline{\tau}} > 0$ such that if $\alpha < \widetilde{\alpha}$ then $\frac{\partial gdp}{\partial\Gamma} > 0$.

12.1. Optimal Fiscal Policy

The Hamiltonian of problem (36) is as follows:

$$H = \frac{\left(A\left((1-\tau(t))h(t)\right)^{\alpha}(1-h(t)-s(t))^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}e^{-(\rho-n)t} +$$
$$\lambda(t)e^{-(\rho-n)t}\left[\left(\tau(t)h(t)-\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{\xi}s(t)^{1-\xi}-bh(t)\right]$$

The first order conditions of optimization problem (36) are as follows:

$$\frac{\alpha\left[c(h(t),\tau(t))\right]^{1-\sigma}}{1-\tau(t)} = \lambda(t)\xi\frac{\dot{h}(t)+bh(t)}{h_h(h,\tau)}\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)} \tag{62}$$

$$\frac{(1-\alpha)\left[c(h(t),\tau(t))\right]^{1-\sigma}}{1-h(t)-s(t)} = \lambda(t)(1-\xi)\frac{\dot{h}(t)+bh(t)}{s(t)} \tag{63}$$

$$\dot{\lambda}(t)-(\rho-n)\lambda(t) =$$
$$-\left[c(h(t),\tau(t))\right]^{1-\sigma}\left[\frac{\alpha}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right]-\lambda(t)\left[\xi\frac{\dot{h}(t)+bh(t)}{h_h(h,\tau)}\tau(t)-b\right] \tag{64}$$

where $\dot{h}(t) + bh(t) = (h_h(h(t),\tau(t)))^{\xi}\, s(t)^{1-\xi}$ and $h_h(h,\tau) = \tau h - \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}$. Using (62) and (63), it yields:

$$s(t) = \frac{\frac{\alpha}{1-\alpha} h_h(h(t),\tau(t)) \frac{\tau(t)}{1-\tau(t)} (1-h(t))}{\frac{\alpha}{1-\alpha} h_h(h(t),\tau(t)) \frac{\tau(t)}{1-\tau(t)} + \frac{\xi}{1-\xi}\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \tag{65}$$

Using (37) and (65), it yields:

$$\lambda(t) = \frac{\left(\frac{1}{A}\right)^{\sigma-1} \frac{\alpha}{\xi} \frac{\tau(t)^{\xi}}{(1-\tau(t))^{\xi+(\sigma-1)\alpha}} \left(\frac{1-\alpha}{\alpha}\right)^{1-\xi} \left(\frac{1-\xi}{\xi}\right)^{(\sigma-1)(1-\alpha)}}{\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1+(\sigma-1)(1-\alpha)}} \times$$
$$\left(\frac{\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)} + \frac{\xi}{1-\xi}\right] h_h(h(t),\tau(t)) - \frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}}{(1-h(t))}\right)^{1-\xi+(\sigma-1)(1-\alpha)} \left(\frac{1}{h(t)}\right)^{\alpha(\sigma-1)} \tag{66}$$

By differentiating this equation with respect to time, it follows that:

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \Omega_1(h(t),\tau(t)) \frac{\dot{\tau}(t)}{\tau(t)}$$
$$- \left[\frac{((1-\xi+(\sigma-1)(1-\alpha)))\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)} + \frac{\xi}{1-\xi}\right] h_h(h(t),\tau(t)) - \frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} + \frac{\xi \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\tau}{\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right) h_h(h,\tau)} \right.$$
$$\left. + \left[(\sigma-1) - \frac{\xi}{h_h(h,\tau)} \frac{\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)} \frac{(1-\tau)}{\alpha}\right] \left[\frac{\alpha}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right] - \frac{1-\xi}{(1-h(t))} \right] \dot{h} \tag{67}$$

where

$$\Omega_1(h,\tau) = \frac{\frac{\partial\lambda}{\partial\tau}}{\lambda(t)} = \frac{\xi+(\sigma-1)\alpha}{1-\tau} +$$
$$\left[\frac{\Omega_2(h,\tau)\left(\frac{1-\gamma}{\gamma}\frac{1}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\left(h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)} + \frac{\xi}{1-\xi}\right] h_h(h(t),\tau(t)) - \frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \right.$$
$$\left. + (1-\xi+(\sigma-1)(1-\alpha)) \frac{\left[\frac{\alpha}{1-\alpha}\frac{1}{(1-\tau(t))^2}\right]\left[h_h(h(t),\tau(t))\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)} + \frac{\xi}{1-\xi}\right] h_h(h(t),\tau(t)) - \frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}} \right] \tau > 0$$

$$\Omega_2(h,\tau) =$$
$$\left[(1+(\sigma-1)(1-\alpha))\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\right] + \frac{\xi^2}{1-\xi}\right]\left[h_h(h(t),\tau(t)) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right] +$$
$$\left[(1\text{-}\xi + (\sigma\text{-}1)(1\text{-}\alpha)) \frac{\alpha\gamma}{1-\alpha}\left(\frac{1-\tau(t)(1-\tau(t))}{(1-\tau(t))^2}\right) + (1+(\sigma\text{-}1)(1\text{-}\alpha))\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1\text{-}\tau(t)}\right]\right] \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}$$

Using (39), (66) and (67), it yields the dynamic system that determine the dynamic behavior of the economy

$$\dot{h}(t) = \left(h_h(h(t),\tau(t))\right)^{\xi}\left(s(h(t),\tau(t))\right)^{1-\xi} - bh(t) \tag{68}$$

$$\begin{aligned}
&\Omega_1(h(t),\tau(t))\frac{\dot{\tau}(t)}{\tau(t)} = \\
&\left[\frac{((1-\xi+(\sigma-1)(1-\alpha)))\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]\tau}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} + \frac{\xi\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\tau}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)h_h(h,\tau)}\right. \\
&\left. + \left[(\sigma\text{-}1) - \frac{\xi}{h_h(h,\tau)}\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)}\frac{(1-\tau)}{\alpha}\right]\left[\frac{\alpha}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right] - \frac{1-\xi}{(1-h(t))}\right]F^h(h(t),\tau(t)) \\
&-\xi\frac{bh(t)}{h_h(h,\tau)}\left[\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\alpha\frac{\tau(t)}{(1-\tau(t))}}\left[\frac{\alpha}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right] + \tau(t)\right] + \rho + m
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
F^h(h,\tau) &= \left(h_h(h,\tau)\right)^{\xi}\left(s(h,\tau)\right)^{1-\xi} - bh \\
s(h,\tau) &= \frac{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau} + \frac{\xi}{1-\xi}\left(h_h(h,\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}
\end{aligned}$$

Proof Proposition 8

It follows from the dynamic system (68) and (69) that at the steady state the following two equation should hold simultaneously:

$$G^{hss}(h,\tau) = \frac{(h_h(h,\tau))^{\xi}}{h}\left(\frac{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}\right)^{1-\xi} - b = 0 \tag{70}$$

$$G^{\tau ss}(h,\tau) = \frac{\xi bh}{h_h(h,\tau)}\left[\tau + \left[\frac{\alpha}{h} - \frac{1\text{-}\alpha}{1\text{-}h\text{-}s(h,\tau)}\right]\frac{1}{\alpha}\frac{\left(h_h(h,\tau)\text{-}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}}\right] - (\rho+m) = 0 \tag{71}$$

Remark 10 *It follows from the above equations that the steady state values do not depend on σ.*

Lemma 11 *There is a function $h^{\dot{h}=0}(\tau)$ such that $G^{hss}(h^{\dot{h}=0}(\tau),\tau) = 0$ and $\frac{\partial h^{\dot{h}=0}(\tau)}{\partial\tau} > 0$.*

Proof. Note that:

$$\frac{\partial\left(\frac{(h_h(h,\tau))^\xi}{h}\right)}{\partial h} = \frac{(h_h(h,\tau))^\xi}{h^2}\left(\xi\frac{\tau h}{h_h(h,\tau)}-1\right) < \frac{(h_h(h,\tau))^\xi}{h^2}\left(\xi\frac{\frac{1}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}-1\right) =$$

$$= \frac{(h_h(h,\tau))^\xi}{h^2}\left(\frac{\xi}{1-\gamma}-1\right) \leq 0 \tag{72}$$

where in the first inequality we use the fact that $\tau h \geq \frac{1}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \Leftrightarrow h \geq \left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{\tau^{\frac{1-\gamma}{\gamma}}}{\gamma}$ and the fact that the function $\frac{h}{h_h(h,\tau)}$ is decreasing[8] in h, and in the last inequality we have use the assumption that $\xi \leq 1-\gamma$. Furthermore:

$$\frac{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} = \frac{\frac{\alpha}{1-\alpha}\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(1-\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)} \tag{73}$$

This function is clearly decreasing in h. Therefore, it follows from (70), (72) and (73) that:

$$\frac{\partial G^{hss}(h,\tau)}{\partial h} < 0$$

Now we analyze the derivative of $G^{hss}(h,\tau)$ with respect to s:

$$\frac{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} = \frac{\frac{\alpha}{1-\alpha}(1-h)}{\frac{\alpha}{1-\alpha}+\frac{\xi}{1-\xi}\frac{\left(1-\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)}{\frac{\tau}{1-\tau}}}$$

Note that:

$$\frac{\partial\left(\frac{\tau^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)}{\partial\tau} = \frac{\frac{1}{\gamma}\tau^{\frac{1}{\gamma}}h_h(h,\tau)-\tau^{\frac{1}{\gamma}}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau\left(h_h(h,\tau)\right)^2} = \frac{\frac{1-\gamma}{\gamma}\tau^{\frac{1}{\gamma}}\left(h_h(h,\tau)+\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau\left(h_h(h,\tau)\right)^2} > 0$$

Therefore $G^{hss}(h,\tau)$ is clearly increasing in τ:

$$\frac{\partial G^{hss}(h,\tau)}{\partial\tau} > 0$$

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[8] $\frac{\partial\left(\frac{h}{h_h(h,\tau)}\right)}{\partial h} = \frac{1}{h_h(h,\tau)}\left[1-\frac{\tau h}{\tau h-\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}\right] < 1$

Thus, it is possible to define $h^{\dot{h}=0}(\tau) \Leftrightarrow G^{hss}(h^{\dot{h}=0}(\tau),\tau) = 0$. It follows from the Implicit function Theorem that:

$$\frac{\partial h^{\dot{h}=0}(\tau)}{\partial \tau} = -\frac{\overbrace{\frac{\partial G^{hss}(h,\tau)}{\partial \tau}}^{\oplus}}{\underbrace{\frac{\partial G^{hss}(h,\tau)}{\partial h}}_{\ominus}} > 0$$

■

The following lemma will prove a statement that is related with $G^{\tau ss}(h,\tau)$.

Lemma 12 $\lim\limits_{\tau\to 0} \frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$

Proof. There are two possible cases:
1) If $\liminf\limits_{\tau\to 0} \frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)} > 0$ then $\lim\limits_{\tau\to 0} \frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$.
2) If $\liminf\limits_{\tau\to 0} \frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)} = 0$. Then there is a sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to+\infty} \tau_i = 0$ and $\lim\limits_{i\to+\infty} \frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau_i) - \frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau_i),\tau_i)} = 0$. Then

$$\lim_{i\to+\infty} \frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau_i) - \frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau_i),\tau_i)} = \lim_{i\to+\infty} \frac{\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau_i h^{\dot{h}=0}(\tau_i) - \left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}} =$$

$$\lim_{i\to+\infty} \frac{\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau_i h^{\dot{h}=0}(\tau_i)(1-\gamma) + \gamma\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} =$$

$$\lim_{i\to+\infty} \frac{\left(1 - \frac{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\tau_i h^{\dot{h}=0}(\tau_i)}\right)}{1-\gamma+\gamma\left(1 - \frac{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\tau_i h^{\dot{h}=0}(\tau_i)}\right)} = 0 \Rightarrow \tag{74}$$

$$\lim_{i\to+\infty} \frac{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}{h^{\dot{h}=0}(\tau_i)} = 1 \Rightarrow \tag{75}$$

Note that:

$$\lim_{i\to+\infty}\left(\frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau)\right)^{\xi}}{bh^{\dot{h}=0}(\tau)}\right)^{\frac{1}{1-\xi}} = \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i h^{\dot{h}=0}(\tau_i)-\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{h^{\dot{h}=0}(\tau_i)}\right)^{\xi}\left(h^{\dot{h}=0}(\tau_i)\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}-\frac{\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}{\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}\right)^{\xi}\left(\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}\right)^{1-\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i 1-\frac{\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}{1}\right)^{\xi}(1)^{1-\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}-\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}\right)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{1}{\tau_i}}\right)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}(1-\gamma)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}\tau_i^{\frac{\xi}{1-\xi}-\frac{1-\gamma}{\gamma}}$$

$$= \lim_{i\to+\infty}\frac{\left(\frac{1}{b}(1-\gamma)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}}{(\tau_i)^{\frac{1-\gamma-\xi}{\gamma}}} = +\infty$$

where in the third inequality we use (70). Thus, it follows from equation (70) that:

$$\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)} =$$

$$\frac{1-\xi}{\xi}\frac{\alpha}{1-\alpha}\left[\left(\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)\right)^{\xi}}{bh^{\dot{h}=0}(\tau)}\right)^{\frac{1}{1-\xi}}\left(1-h^{\dot{h}=0}(\tau)\right)-1\right] \Rightarrow$$

$$\lim_{i\to+\infty}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)} = +\infty$$

where we used equation (75) in the forth equality. Thus in the two possible cases mentioned above:

$$\lim_{\tau\to 0}\frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$$

■

Now we proceed to prove proposition 8:

Proof. At steady state the following equation should hold (see equations (70), (71), and lemma 11):

$$G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) =$$
$$\xi b\left[\frac{\tau h^{\dot{h}=0}(\tau)}{h_h(h^{\dot{h}=0}(\tau),\tau)}+\left[\alpha-\frac{(1-\alpha)\,h^{\dot{h}=0}(\tau)}{1-h^{\dot{h}=0}(\tau)-s(h^{\dot{h}=0}(\tau),\tau)}\right]\frac{1}{\alpha}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)}\right]$$
$$-(\rho+m) = 0$$

Let us define $f(\tau)=\xi b\left[\frac{\tau h^{\dot{h}=0}(\tau)}{h_h(h^{\dot{h}=0}(\tau),\tau)}+\left[\alpha-\frac{(1-\alpha)h^{\dot{h}=0}(\tau)}{1-h^{\dot{h}=0}(\tau)-s(h^{\dot{h}=0}(\tau),\tau)}\right]\frac{1}{\alpha}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)}\right]$. Note that $G^{hss}(1,\tau) = -b$ (see eq. 70), therefore $h^{\dot{h}=0}(\tau) < 1$. Thus, $f(\tau)$ is a continuous function for $\tau\in(0,1]$. Furthermore, it follows from lemma 12 that $\lim_{\tau\to 0}f(\tau) = +\infty$. Thus, if $(\rho+m)\geq\min_{\tau\in[0,1]}f(\tau)$ then there is at least a τ^{ss} such that $f(\tau^{ss}) = \rho+m$. Since $\lim_{\tau\to 0}\frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$, $f(\tau)$ is strictly decreasing in an interval $(0,\hat{\tau})$ where $\hat{\tau}>0$.

Thus, if $(\rho+m) > \max\limits_{\tau\in[\hat{\tau},1]} f(\tau)$, then $f(\tau^{ss}) = \rho+m$ for a $\tau^{ss} \in (0,\hat{\tau})$. Since in such interval $f(\tau)$ is strictly decreasing, there is a unique τ^{ss} such that $f(\tau^{ss}) = \rho+m$. ∎

Proof Proposition 4

It is easy to check that $F^h(h,\tau)$ is increasing in τ and decreasing in h, thus, it follows from the Implicit Function Theorem that the locus $\dot{h}(t) = 0$ has a positive slope:

$$\tau_{\dot{h}=0}(h) \overset{def}{\Leftrightarrow} F^h\left(h, \tau_{\dot{h}=0}(h)\right) = 0; \quad \frac{\partial \tau_{\dot{h}=0}(h)}{\partial h} = -\frac{\frac{\partial F^h(h,\tau)}{\partial h}}{\frac{\partial F^h(h,\tau)}{\partial \tau}} > 0$$

Let us define $F^\tau(h,\tau)$ as the function that determine $\dot{\tau}(t)$ (see equation 69)

$$\begin{aligned} F^\tau(h,\tau) &= \frac{\tau}{\Omega_1(h,\tau)} \times \\ &\Bigg\{ \Bigg[\left(\frac{((1-\xi+(\sigma-1)(1-\alpha)))\left[\frac{\alpha}{1-\alpha}\frac{\tau}{1-\tau}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left(\left[\frac{\alpha}{1-\alpha}\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\right]h_h(h,\tau)-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} + \frac{\xi\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)} \right) \frac{\tau}{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \\ &+ \left((\sigma\text{-}1) - \frac{\xi}{h_h(h,\tau)} \frac{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau} \frac{(1-\tau)}{\alpha} \right) \left[\frac{\alpha}{h} - \frac{1-\alpha}{1-h-s(h,\tau)}\right] - \frac{1-\xi}{(1-h)} \Bigg] F^h(h,\tau) \\ &- \xi\frac{bh}{h_h(h,\tau)} \left[\frac{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\alpha\frac{\tau}{(1-\tau)}} \left[\frac{\alpha}{h} - \frac{1-\alpha}{1-h-s}\right] + \tau \right] + \rho + m \Bigg\} \end{aligned} \tag{76}$$

Lemma 13 *In a surrounding of h^{ss} if $h < h^{ss}$ then $\tau_{\dot{h}=0}(h) < \tau_{\dot{\tau}=0}(h)$ and if $h > h^{ss}$ then $\tau_{\dot{h}=0}(h) > \tau_{\dot{\tau}=0}(h)$. Furthermore, if $\tau(t) < \tau_{\dot{\tau}=0}(h(t))$ then $\tau_{\dot{\tau}=0}(h(t)) > 0$ and if $\tau(t) > \tau_{\dot{\tau}=0}(h(t))$ then $\tau_{\dot{\tau}=0}(h(t)) > 0$. The locus $\tau_{\dot{\tau}=0}(h(t))$ has positive slope($\frac{\partial \tau_{\dot{h}=0}(h)}{\partial h} > 0$). Finally, $\frac{\partial F^\tau(h^{ss},\tau^{ss})}{\partial \tau} > 0$.*

Proof. Consider a point (h,τ) in the locus $\dot{h} = 0$ ($\tau = \tau_{\dot{h}=0}(h)$), it follows from (69) and the definition of $F^h(h,\tau)$ that:

$$F^\tau\left(h, \tau_{\dot{h}=0}(h)\right) \frac{\Omega_1(h, \tau_{\dot{h}=0}(h))}{\tau_{\dot{h}=0}(h)} = -G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) = -G^{\tau ss}(h^{\dot{h}=0}(\tau), \tau) \tag{77}$$

We prove already in the proof of proposition 8 that:

$$\begin{aligned} G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) > 0 \text{ if } \tau < \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) > 0 \text{ if } h < h^{ss} \\ G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) = 0 \text{ if } \tau = \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) = 0 \text{ if } h = h^{ss} \\ G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) < 0 \text{ if } \tau > \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) < 0 \text{ if } h > h^{ss} \end{aligned}$$

where we have used the fact that $h^{\dot{h}=0}(\tau)$ a strictly increasing and therefore bijective function. Given equation (77, this implies that:

$$\begin{array}{l} F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) < 0 \text{ if } h < h^{ss} \\ F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) = 0 \text{ if } h = h^{ss} \\ F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) > 0 \text{ if } h > h^{ss} \end{array}$$

Note that $\lim_{\tau \to \left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}} \frac{1}{h_h(h,\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}} = +\infty$. It follows from (69) that when $h < h^{ss}$, $\dot{\tau}(t) > 0$ for τ close enough to $\left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}$. Thus, when $h < h^{ss}$ the locus $\dot{\tau}(t) = 0$ is in between $\tau_{\dot{h}=0}(h)$ and $\left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}$. Furthermore when $\tau < \tau_{\dot{\tau}=0}(h)$ then $\dot{\tau}(h,\tau) < 0$ and if $\tau > \tau_{\dot{\tau}=0}(h)$ then $\dot{\tau}(h,\tau) > 0$. This implies that $\frac{\partial F^{\tau}(h, \tau_{\dot{\tau}=0}(h))}{\partial \tau} > 0$. ■

Now we proceed to prove proposition 9:

Proof. The dynamic system in a surrounding of the steady state may be linearized as follows:

$$\left[\begin{array}{c} \dot{h}(t) \\ \dot{\tau}(t) \end{array}\right] = \left[\begin{array}{cc} \frac{\partial F_h(h^{ss}, s^{ss})}{\partial h} & \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau} \\ \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h} & \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} \end{array}\right] \left[\begin{array}{c} h(t) - h^{ss} \\ \tau(t) - \tau^{ss} \end{array}\right]$$

The eigenvalues are as follows:

$$\left|\begin{array}{cc} \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h} - \lambda & \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau} \\ \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h} & \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} - \lambda \end{array}\right| = \lambda^2 - tr\,\lambda + Det = 0 \Rightarrow \lambda = \frac{tr \pm \sqrt{tr^2 - 4Det}}{2}$$

It follows from lemma 13:

$$\begin{aligned} tr &= \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h} + \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} > 0 \\ Det &= \underbrace{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h}}_{-} \underbrace{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau}}_{+} - \underbrace{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau}}_{+} \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h} = \\ Det &= \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau} \left[\frac{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h}}{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau}} - \frac{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h}}{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau}}\right] = \\ Det &= \underbrace{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau}}_{+} \underbrace{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau}}_{+} \underbrace{\left[\frac{\partial \tau_{\dot{\tau}=0}(h^{ss})}{\partial h} - \frac{\partial \tau_{\dot{h}=0}(h^{ss})}{\partial h}\right]}_{-} < 0 \end{aligned}$$

Thus, one of the eigenvalues is positive and the other is negative. This means that the steady state is a saddle point. ■