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Optimal Taxation with Multiple Dimensions of Heterogeneity

Katy Bergstrom
William Dodds

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Abstract

This paper develops a general theory of optimal income taxation with multiple dimensions of agent heterogeneity. The main technical hurdle in developing this theory is the possibility that individuals have multiple optimal incomes. Using a perturbation approach, optimal tax formulas are derived that account for the possibility that individuals have multiple optima and, hence, account for the possibility that individuals jump between their optimal income levels when the tax schedule is perturbed. The magnitude of these effects is quantified, thereby augmenting the optimal tax formulas from Saez (2001) with additional "jumping effect" terms. The paper provides a partial characterization of when individuals with multiple optimal incomes may exist under the optimal tax schedule. Finally, the paper derives a new methodology to simulate optimal income tax schedules with multidimensional heterogeneity. This method is implemented numerically, showing that individuals with multiple optimal income levels can exist under the optimal tax schedule.

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Optimal Taxation with Multiple Dimensions of Heterogeneity*

Katy Bergstrom[†] and William Dodds[‡]

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[†]Development Research Group, World Bank. Email: kbergstrom@worldbank.org.

[‡]Charles River Associates. Email: williamcdodds@gmail.com.

1 Introduction

The canonical Mirrlees (1971) optimal income taxation problem examines how to best redistribute labor income among a population of individuals who differ only in terms of how productive they are. However, in reality, individuals differ on many dimensions such as preferences for consumption relative to leisure, labor supply elasticities, and participation costs of being in the labor force. Understanding how the results from the Mirrlees optimal tax model generalize to settings with richer individual heterogeneity is thus an important agenda for both economists and policy-makers.

The fundamental challenge with extending the results from Mirrlees's optimal tax problem to settings where individuals differ on many (unobservable) dimensions is the possibility that some individuals have multiple optimal income levels under the optimal tax schedule.[1] In the approach developed by Piketty (1997), Diamond (1998), and Saez (2001), optimal income tax schedules are derived by considering small variations to the tax schedule. However, if some individuals have multiple optimal income levels, then these individuals will not respond smoothly to a small variation in the tax schedule; instead, they will "jump" between their initially optimal income levels, which complicates the analysis of optimal taxation. Typically the optimal income taxation literature (both in settings with unidimensional agent heterogeneity and with multidimensional agent heterogeneity) has assumed away the presence of individuals with multiple optimal income levels.[2] While we will show that, under standard assumptions, one can rule out the possibility that individuals have multiple optimal income levels when they differ only in terms of how productive they are, we show that it is, in general, impossible to rule out the presence of individuals with multiple optima when heterogeneity is multidimensional.

The first part of this paper develops a general theory of optimal income taxation with multidimensional agent heterogeneity that accounts for the possibility that individuals have multiple optimal income levels. We consider a population of individuals who differ on many arbitrary dimensions and choose labor supply to maximize their own utility. The government chooses an income tax schedule to maximize the total welfare of this population taking into account agents' behavioral responses to tax changes.[3] To derive a formula for the optimal tax schedule, we analyze the welfare impacts of a novel tax

[1]This paper will only be concerned with agent heterogeneity that is unobservable to the government, or equivalently, heterogeneity that the government does not wish to condition the tax schedule on. This is in direct contrast to the tagging literature (e.g., Akerlof (1978), Cremer et al. (2010), or Mankiw and Weinzierl (2010)) whereby the government bases taxes on observable characteristics.

[2]To the best of our knowledge, we do not know of any papers that account for the possibility that some individuals have multiple optimal income levels in settings with multidimensional agent heterogeneity. However, using a mechanism design approach, Hellwig (2010) explores the possibility that individuals have multiple optimal income levels in a setting with unidimensional heterogeneity.

[3]Thus, both the action space of the agents and the policy space of the government are unidimensional. This is in contrast to papers such as Kleven et al. (2009) or Golosov et al. (2014), which consider multidimensional action and policy spaces.

perturbation. Our perturbation approach applies insights from Golosov et al. (2014) and Jacquet et al. (2013) to mathematically formalize the perturbation approach from Saez (2001) while retaining Saez's (2001) core economic intuition. We allow for the possibility that individuals have multiple optimal income levels and, thus, jump between these optimal income levels in response to our perturbation. We quantify the magnitude of these jumping effects and show that the optimal tax formula consists of multidimensional versions of the mechanical, elasticity, and income effect terms originally discussed in Saez (2001) as well as additional jumping effect terms.

We then derive a partial characterization of when individuals will have multiple optimal income levels (or, equivalently, when jumping effects will occur). Our first result is that when individuals only differ in terms of how productive they are, no individual will ever have multiple optimal income levels under the optimal tax schedule given continuity assumptions on the productivity distribution and sensible assumptions on individual utility. We then use this result to show that certain classes of problems with multiple dimensions of heterogeneity will also never yield an individual with multiple optima under the optimal tax schedule. Our second result, however, proves the existence of sensible conditions for which some individuals *will* have multiple optimal income levels under the optimal tax schedule when heterogeneity is multidimensional.

The second part of this paper develops and implements a new method to numerically simulate optimal income tax schedules when agents have multiple dimensions of heterogeneity and some individuals (potentially) have multiple optimal incomes. Our key contribution is showing we can differentiate the governments optimality condition to yield a computable differential equation that characterizes the optimal tax schedule.[4] This differential equation holds at all income levels for which no individual with multiple optima locates. If no individuals have multiple optimal incomes, we can simply solve this differential equation to yield the optimal tax schedule.[5] If, on the other hand, individuals do have multiple optimal incomes, we prove that the optimal tax schedule will be non-differentiable at these income levels and that there will be bunching.[6] Hence, we augment our simulation procedure to solve the resulting piece-wise differential equation, searching over the individual who has multiple optimal incomes and the size of the discontinuities in the marginal tax rates at their optimal incomes. We then consider a

[4]This is useful because it yields an expression for the optimal marginal tax rate at a given income level that is not a function of the optimal marginal tax rate at any other income level. This is in contrast to the optimality condition from, for example, Saez (2001) in which the optimal marginal tax rate at a given income level is a function of the marginal tax rates at incomes above the given income level.

[5]We show numerically that in unidimensional problems, our method yields identical tax schedules to the Hamiltonian approach a la Mirrlees (1971) or Saez (2001).

[6]Our optimal tax formulas also allow for the possibility of bunching at non-differentiable points of the tax schedule. While bunching has been largely assumed away in the optimal tax literature using the tax perturbation approach, bunching has been explored extensively using mechanism design approaches, see, for example, Lollivier and Rochet (1983), Guesnerie and Laffont (1984), and Ebert (1992).

numerical example where individuals differ not only in productivity but also in curvature of disutility over labor (which is inversely proportional to the elasticity of earnings with respect to the tax rate). We illustrate that our method can solve for tax schedules when agent heterogeneity is multidimensional and is able to handle the possibility that some individuals have multiple optimal income levels.

We are not the first paper to explore optimal taxation with multidimensional agent heterogeneity. Assuming that individuals have a single optimal income level, Mirrlees (1986) derives optimality conditions using a mechanism design approach, although the equations prove to be unwieldy and computationally intractable. Saez (2001) conjectures that the optimal tax formulas he derives can be extended to multiple dimensions of heterogeneity if one simply averages the relevant labor supply elasticities. Jacquet and Lehmann (2020) show formally that this conjecture holds under the assumption that no individuals have multiple optimal income levels.[7] [8] Our primary contributions are (1) proving that individuals can have multiple optima under sensible conditions when agent heterogeneity is multidimensional, and (2) showing that by quantifying and accounting for jumping effects, which arise when individuals have multiple optima, we can characterize the optimal tax schedule in multidimensional settings.

While the presence of individuals with multiple optima has typically been assumed away in the optimal tax literature, Hellwig (2010) explores this possibility using a mechanism design approach in the context of a general unidimensional incentive problem, proving the existence of an agent with multiple optima when there are mass points in the type distribution.[9] In contrast, we use a tax perturbation approach to quantify the jumping effects that result from individuals with multiple optimal incomes and show that individuals with multiple optima can exist with smooth type distributions if agent heterogeneity is multidimensional.

Finally, prior work has utilized a number of different approaches to simulate optimal income tax schedules. Mirrlees (1971) showed that one can solve the unidimensional optimal income tax problem using a system of equations derived from Hamiltonian optimization. Mirrlees's Hamiltonian approach has been used not only in optimal income taxation papers with unidimensional heterogeneity, e.g., Saez (2001), but also in papers

[7] Scheuer and Werning (2016) also note in an appendix that one can average the relevant elasticities to generalize the optimal tax formula to account for multidimensional heterogeneity if no individuals have multiple optimal income levels.

[8] Additionally, a number of papers, such as Boadway et al. (2002), Choné and Laroque (2010), and Lockwood and Weinzierl (2016), have explored how adding various specific forms of heterogeneity impacts different aspects of the tax schedule.

[9] Moreover, the literature on optimal taxation with extensive margin effects also allows for the possibility that individuals have multiple optima; however, individuals can only be indifferent between working and non-working (e.g., see Jacquet et al. (2013), Choné and Laroque (2005), Saez (2002), and Diamond (1980)). Our paper is, to our knowledge, the first to explicitly take into account the possibility that individuals have two non-zero optimal incomes in a multidimensional setting.

with multidimensional agent heterogeneity where labor supply decisions can be rewritten as a function of a unidimensional parameter (e.g., Choné and Laroque (2010) and Lockwood and Weinzierl (2016)). Alternatively, other papers have used iterative methods to solve for the optimal tax schedule in both unidimensional and multidimensional settings (e.g., Mankiw et al. (2009) or Jacquet and Lehmann (2020)).[10] However, the Hamiltonian approach does not appear to be feasible with multiple, arbitrary dimensions of heterogeneity (given the number of incentive compatibility constraints) and we found the iterative method impossible to adapt to the situation in which some individuals have multiple optimal income levels due to numerical instability. Thus, we contribute by developing a novel simulation method that allows for agents to differ on many, arbitrary dimensions and allows for the possibility that some individuals have multiple optimal income levels.

The rest of the paper proceeds as follows: Section 2 states the optimal taxation problem we consider and discusses elasticity concepts as well as important assumptions, Section 3 derives first order conditions for the optimal schedule with multidimensional heterogeneity, Section 4 discusses our results about the existence (and non-existence) of individuals with multiple optimal income levels, Section 5 develops our simulation methodology and presents the results of our simulations, and Section 6 concludes.

2 The Mirrlees Optimal Tax Problem with Multiple Dimensions of Heterogeneity

In this section we discuss the setup of the Mirrlees optimal taxation problem when individuals have multiple dimensions of heterogeneity. We present the government problem, individual preferences, as well as a number of elasticity concepts that will be necessary to employ the perturbation argument.

2.1 Preferences

The model consists of a population of individuals, indexed by a productivity level $n \in N$ with $N \subseteq \mathbb{R}_+$ and a second type of heterogeneity $\alpha \in A$ (note, α could represent a vector of characteristics). We assume the distribution of productivities is continuous, while the distribution of α is discrete. We assume α comes from a discrete distribution for simplicity and to match our simulation procedure.[11] Denote the joint CDF of types by

[10] Other papers, such as Judd et al. (2018), have explored optimal taxation with multidimensional heterogeneity with discrete types using brute force numerical optimization.

[11] A continuous α distribution merely complicates the exposition. Appendix B derives optimality conditions for a continuous α distribution. As will be seen later, the key difference is that when α

$F(n, \alpha)$, the conditional CDF of n given α as $F(n|\alpha)$, and probability mass function for α as $p(\alpha)$. We assume the conditional distribution $F(n|\alpha)$ is continuously differentiable $\forall$ $\alpha \in A$. Individuals have utility over consumption, denoted c, and labor supply, denoted l, according to a utility function that varies with α: $u(c, l; \alpha)$, with $u_c > 0, u_l < 0, u_{cc} \leq 0, u_{ll} < 0$.[12] Individuals are able to transform their labor supply into income, denoted z, with a simple linear production function: $z = nl$. If we denote the tax function as $T(z)$, we have that $c(z) \equiv z - T(z)$, which allows us to write the individual maximization problem as:

$$\max_z \ u\left(c(z), \frac{z}{n}; \alpha\right)$$

Note, even when we express the utility function in terms of z, we still use the notation $u_c\left(z - T(z), \frac{z}{n}; \alpha\right)$ and $u_l\left(z - T(z), \frac{z}{n}; \alpha\right)$ to refer to the derivatives with respect to the first and second arguments, respectively. Using this simple change of variables allows us to think of people as having preferences over pre-tax income z as well as post-tax income $c(z)$; this formulation will be useful going forward as it simplifies the notation in both the written and graphical exposition.

2.2 Government Problem

The government gets to set the tax function $T(z)$ and seeks to maximize a general social welfare function that sums an increasing transformation of individual utility levels, $W(u; n, \alpha)$, subject to a revenue constraint that total consumption in society plus exogenous government expenditures, E, must not exceed total income.[13] Additionally, the government faces incentive compatibility constraints: individuals will choose income levels to maximize their utility subject to the tax schedule. The government cannot observe n or α for any individual, but can observe the distribution of types $F(n, \alpha)$. We can write

is continuously distributed, the marginal tax schedule can be continuous at incomes for which some individuals have multiple optimal incomes; whereas, when α is discretely distributed, we show that at these income levels, the marginal tax schedule is discontinuous - see Proposition 5.

[12]Labor supply should be interpreted broadly as a composite measure of the long-term total effort a worker exerts; hence, it should be interpreted as a function of hours worked, work intensity as well as human capital investments individuals make in themselves prior to working.

[13]See Jacquet and Lehmann (2020) for further details on this general social welfare function.

the government problem as follows:

$$\max_{T(z)} \sum_{\alpha\in A}\int_0^\infty W\left(u\left(c^*(n,\alpha),\frac{z^*(n,\alpha)}{n};\alpha\right);n,\alpha\right)dF(n|\alpha)p(\alpha)$$
$$\text{s.t.} \sum_{\alpha\in A}\int_0^\infty c^*(n,\alpha)dF(n|\alpha)p(\alpha)+E\leq\sum_{\alpha\in A}\int_0^\infty z^*(n,\alpha)dF(n|\alpha)p(\alpha)$$
$$z^*(n,\alpha)\in \underset{z}{\text{argmax}}\ u\left(z-T(z),\frac{z}{n};\alpha\right)\ \forall n,\alpha$$
$$c^*(n,\alpha)=z^*(n,\alpha)-T\left(z^*(n,\alpha)\right)$$

Note, for ease of notation, we have omitted that agents' optimal choice of income, z^*, will be a function of the tax schedule. In order to characterize the optimal tax schedule, we will use a variational approach as in Saez (2001). To start with, we write down the government's Lagrangian as follows, noting that $z^*(n,\alpha)$ satisfies agents' incentive compatibility constraints:

$$\mathcal{L}=\sum_{\alpha\in A}\int_0^\infty\left[W\left(u\left(c^*(n,\alpha),\frac{z^*(n,\alpha)}{n};\alpha\right);n,\alpha\right)+\lambda T\left(z^*(n,\alpha)\right)\right]dF(n|\alpha)p(\alpha)$$

where, without loss of generality, we set $E=0$, and have used $T(z^*(n,\alpha))=z^*(n,\alpha)-c^*(n,\alpha)$. The variational argument that we employ will maximize this Lagrangian, taking into account the incentive compatibility constraints (i.e., taking into account that $z^*(n,\alpha)$ varies with the tax schedule). However, we next take a slight detour to discuss technical assumptions and to define a number of elasticities that will be useful towards understanding how changing the tax schedule affects the government Lagrangian.

2.3 Technical Assumptions

In order for us to employ a variational argument to derive the optimal tax schedule, we need to make the following standard assumption on individual preferences:

Assumption 1. *The Single Crossing Property (SCP) holds:* $-\frac{u_l\left(c,\frac{z}{n};\alpha\right)}{nu_c\left(c,\frac{z}{n};\alpha\right)}$ *is decreasing in* n $\forall c,z,\alpha$.

Assumption 1 ensures that individual preferences satisfy a kind of monotonicity: for a given α, the steepness of indifference curves in (income, consumption) space is increasing in productivity n. Sufficient conditions for (SCP) to hold are that $-\frac{1}{n}u_l\left(c,\frac{z}{n};\alpha\right)$ is strictly decreasing in n (i.e., the utility cost of labor is convex) and that $u_c\left(c,\frac{z}{n};\alpha\right)$ is weakly increasing in n (i.e., leisure and consumption are complements). We will use Assumption 1 primarily to ensure that income exhibits monotonicity in productivity n:

Lemma 1. *If (SCP) holds, then* $z^*(n,\alpha)$ *is non-decreasing in productivity* n $\forall\ \alpha$. *More-*

over, if (SCP) holds, $z^*(n,\alpha)$ is increasing in n $\forall$ α whenever $T'(z)$ exists.

Proof. See Appendix A.1. □

With Lemma 1, we can prove the following Lemma:

Lemma 2. *If (SCP) holds, the set of individuals with multiple optimal income levels is countable.*

Proof. See Appendix A.2. □

Graphically, the main idea of the proof is illustrated in Figure 1, which depicts an impossible scenario under (SCP). Importantly, under (SCP), indifference curves for two individuals with the same α cannot cross more than once. Thus, if individual $(n_1, \bar{\alpha})$ has two optima (illustrated by the blue indifference curve), then no individual $(n_2, \bar{\alpha})$ can optimally locate in between his two optima (illustrated by the red indifference curve crossing the blue indifference curve twice). Thus, for fixed α, each individual with multiple optima can be associated with a unique jump discontinuity of $z^*(n,\alpha)$. Because $z^*(n,\alpha)$ is a weakly monotonic function in n for each α by (SCP), it can only have a countable number of jump discontinuities. As such, the number of individuals with multiple optima is countable for a fixed α and, because the union of countable sets is also countable, the number of individuals with multiple optima across all α's is also countable.

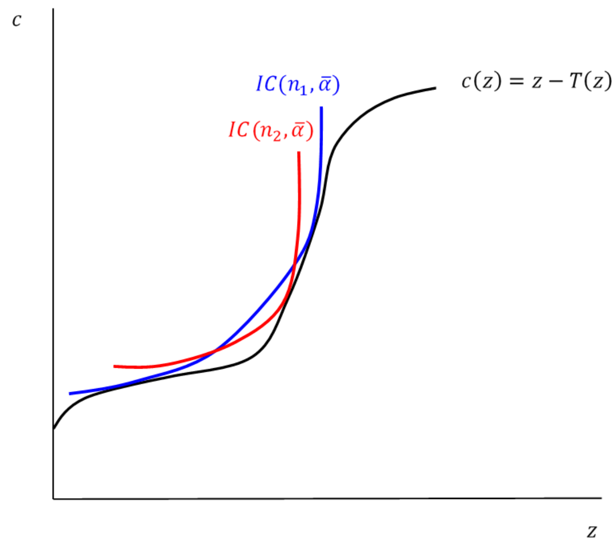

Figure 1: Impossible Scenario Under (SCP)

Next, for each α, denote $m_1(\alpha), m_2(\alpha), \ldots$ as the productivity levels (in ascending order) of individuals with multiple optimal income levels under a given tax schedule T (for ease of notation, we have suppressed that $m_i(\alpha)$ is a function of the tax schedule T). We denote $z_i^{*-}(\alpha)$ and $z_i^{*+}(\alpha)$ as the minimum and maximum optimal income levels for type $(m_i(\alpha), \alpha)$:

$$z_i^{*-}(\alpha) = \min \left(\underset{z}{\operatorname{argmax}}\, u\left(z - T(z), \frac{z}{m_i(\alpha)}; \alpha\right)\right)$$

$$z_i^{*+}(\alpha) = \max \left(\underset{z}{\operatorname{argmax}}\, u\left(z - T(z), \frac{z}{m_i(\alpha)}; \alpha\right)\right)$$

Then $m_i(\alpha)$ satisfies the following indifference condition:

$$u\left(z_i^{*-}(\alpha) - T(z_i^{*-}(\alpha)), \frac{z_i^{*-}(\alpha)}{m_i(\alpha)}; \alpha\right) = u\left(z_i^{*+}(\alpha) - T(z_i^{*+}(\alpha)), \frac{z_i^{*+}(\alpha)}{m_i(\alpha)}; \alpha\right)$$

We now make an assumption that there exists a minimum distance D_1 between the productivity levels of individuals who have multiple optimal income levels. Specifically, we assume:

Assumption 2. *Under the optimal tax schedule, there exists a minimum distance $D_1 > 0$ s.t. $m_i(\alpha) - m_{i-1}(\alpha) > D_1 \ \forall\ \alpha$.*

This assumption is needed to rule out pathological settings whereby all individuals with rational productivity levels have multiple optima, for example. We also make an assumption that there exists a minimum distance D_2 between income levels for which there exists someone with multiple optimal income levels:

Assumption 3. *Define the set of income levels $\{z_i^{mult}\}$ to be the income levels such that there exists an individual with optimal income $z^* \in \{z_i^{mult}\}$ that has other optimal income levels. Under the optimal tax schedule, there exists a minimum distance $D_2 > 0$ s.t. $z_i^{mult} - z_{i-1}^{mult} > D_2$.*

We will also make the following assumption on the shape of the optimal tax schedule:

Assumption 4. *$T(z)$ is "piece-wise smooth": $T(z)$ is continuous $\forall z$ and twice continuously differentiable $\forall z$ except at most for a countable set of kinks, $\{K_i\}$, separated by a minimum distance, $K_i - K_{i-1} > D_3$.*[14]

2.4 Elasticities

Our goal is to analyze the effect on the government Lagrangian of perturbing the tax schedule $T(z)$ in the direction of a twice continuously differentiable schedule $\tau(z)$.[15] Formally, for $\mu \in \mathbb{R}_+$, we define the perturbed tax schedule $\tilde{T}(z) = T(z) + \mu\tau(z)$. We then

[14] We show in Section 4 that if an optimal tax schedule has an individual with multiple optimal income levels, it cannot be twice continuously differentiable at these income levels. Thus, it is insufficient to consider everywhere twice continuously differentiable tax functions when some individuals potentially have multiple optimal income levels.

[15] $\tau(z)$ could, in principle, also be piece-wise smooth, but this additional complication will not be needed to derive our main optimality condition.

derive the change in the government Lagrangian as $\mu \to 0$. In other words, we compute the Gateaux derivative of the government Lagrangian following the local perturbation of the tax schedule $T(z)$ in the direction of $\tau(z)$.[16] However, before doing so, we define two elasticities, the compensated elasticity and the income effect parameter. The compensated elasticity, $Z^c_{n,\alpha}$, captures how type (n, α) changes her optimal income when we decrease her marginal tax rate while leaving her tax liability unchanged; this corresponds to perturbing the tax schedule in the direction of $\tau(z) = -(z - z^*(n, \alpha))$.[17] The income effect, $\eta_{n,\alpha}$, captures how type (n, α) changes her optimal income when we decrease her tax liability while leaving her marginal tax rate unchanged; this corresponds to a perturbation in the direction of $\tau(z) = -1$. First, let's derive an expression for how type $(n, \alpha)'s$ optimal income changes as we move in the direction of any $\tau(z)$.

Under the perturbed tax schedule, whenever $T'(z)$ exists, an agent's optimal income must satisfy her FOC:

$$u_c\left(z^* - T(z^*) - \mu\tau(z^*), \frac{z^*}{n}; \alpha\right)(1 - T'(z^*) - \mu\tau'(z^*)) + \frac{1}{n}u_l\left(z^* - T(z^*) - \mu\tau(z^*), \frac{z^*}{n}\right) = 0$$

Next, we show that if an individual has a unique optimal income located at a point where $T''(z)$ exists and is continuous, this individual's second order condition must hold strictly:
Lemma 3. *An individual's second order condition must hold strictly if the individual has a unique optimal income level and the tax schedule at this unique optimal income level is twice continuously differentiable.*

Proof. See Appendix A.3. □

Thus, for individuals with a unique optimal income located at a point where $T''(z)$ exists and is continuous, we can use the implicit function theorem to describe how their optimal income changes with μ for any $\tau(z)$:[18]

$$\left.\frac{\partial z^*}{\partial \mu}\right|_{\mu=0} = \frac{u_c^*\tau'(z^*) + u_{cc}^*(1 - T'(z^*))\tau(z^*) + \frac{\tau(z^*)}{n}u_{cl}^*}{u_{cc}^*(1 - T'(z^*))^2 + \frac{2}{n}u_{cl}^*(1 - T'(z^*)) + \frac{1}{n^2}u_{ll}^* - u_c^*T''(z^*)} \quad (1)$$

where, in Equation 1, we have omitted that z^* is a function of (n, α) and where $u_c^* = u_c\left(z^*(n, \alpha) - T\left(z^*(n, \alpha)\right), \frac{z^*(n,\alpha)}{n}; \alpha\right)$ denotes the marginal utility of consumption for (n, α) at their optimal income level (u_{cc}^*, u_{cl}^*, and u_{ll}^* are similarly defined).

[16]This approach is used in Golosov et al. (2014), for example.

[17]Conceptually, the compensated elasticity captures the behavioral response of (n, α) to a "rotation in the tax schedule" around her initially optimal income $z^*(n, \alpha)$.

[18]The insight that one can define elasticities taking into account the non-linearity of the tax schedule using the implicit function theorem so long as the second order condition holds strictly comes from Jacquet et al. (2013).

We can now define the compensated elasticity for type (n,α) as follows:[19]

$$\begin{aligned} Z^c_{n,\alpha} &\equiv \frac{1-T'(z^*)}{z^*}\frac{dz^*}{d\mu}\Big|_{\mu=0,\tau(z)=-(z-z^*(n,\alpha))} \\ &= -\frac{1-T'(z^*)}{z^*}\frac{u^*_c}{u^*_{cc}(1-T'(z^*))^2+\frac{2}{n}u^*_{cl}(1-T'(z^*))+\frac{1}{n^2}u^*_{ll}-u^*_cT''(z^*)} \end{aligned}$$

And we can define the income effect parameter for type (n,α) as follows:

$$\begin{aligned} \eta_{n,\alpha} &\equiv (1-T'(z^*))\frac{dz^*}{d\mu}\Big|_{\mu=0,\tau(z)=-1} \\ &= (1-T'(z^*))\frac{-u^*_{cc}(1-T'(z^*))-\frac{1}{n}u^*_{cl}}{u^*_{cc}(1-T'(z^*))^2+\frac{2}{n}u^*_{cl}(1-T'(z^*))+\frac{1}{n^2}u^*_{ll}-u^*_cT''(z^*)} \end{aligned}$$

We can now rewrite how optimal income changes as we move in the direction of any $\tau(z)$ (i.e., rewrite Equation 1) in terms of these elasticities:

$$\frac{\partial z^*}{\partial\mu}\Big|_{\mu=0} = -\frac{Z^c_{n,\alpha}z^*}{1-T'(z^*)}\tau'(z^*) - \frac{\eta_{n,\alpha}}{1-T'(z^*)}\tau(z^*) \tag{2}$$

Finally, note that all of these elasticities are *endogenous* to the tax schedule; hence an individual's elasticity under the optimal tax schedule will be different from her empirical elasticity under the observed tax schedule. Also, note that this formulation defines elasticities under a given, potentially non-linear, tax schedule as in Jacquet et al. (2013), Scheuer and Werning (2017), and Golosov et al. (2014). This is contrasted to the elasticity definitions in Saez (2001), which are defined under a linearized schedule. This will be useful going forward as it enables us to write our first order conditions for the optimal tax schedule as a function of the true earnings density as opposed to the "virtual density" of incomes as in Saez (2001).

3 Optimal Tax Schedule

We will now derive a differential equation that characterizes the optimal tax schedule assuming that all agents have one global optimal income level under the optimal tax schedule. We will then relax this assumption and show how this changes the differential equation characterizing the optimal tax schedule.

[19] Alternatively, we could have defined the uncompensated elasticity as $Z^u_{n,\alpha} \equiv \frac{1-T'(z^*)}{z^*}\frac{dz^*}{d\mu}|_{\mu=0,\tau(z)=-z}$ and defined the compensated elasticity via the Slutsky equation $Z^c_{n,\alpha} = Z^u_{n,\alpha} - \eta_{n,\alpha}$.

3.1 All Individuals Have a Unique Optimal Income

We know that starting from the optimal tax schedule, the derivative of the government Lagrangian in the direction of $\tau(z)$ must be 0. Thus, the optimal schedule must satisfy the following condition:

$$\frac{\partial}{\partial\mu}\left[\sum_{\alpha\in A}\int_0^\infty\left[W\left(u\left(z^*-T(z^*)-\mu\tau(z^*),\frac{z^*}{n};\alpha\right);n,\alpha\right)+\lambda(T(z^*)+\mu\tau(z^*))\right]dF(n|\alpha)p(\alpha)\right]\Bigg|_{\mu=0}=0$$

where, for brevity, we omit that z^* is a function of (n,α) as well as the perturbed tax schedule $T(\cdot)+\mu\tau(\cdot)$. Taking the derivative w.r.t. μ and evaluating at $\mu=0$, the optimal tax schedule must satisfy:

$$\sum_{\alpha\in A}\int_0^\infty\left[-W_u(u^*)u_c^*\tau(z^*)+\lambda\left(\frac{\partial T(z^*)}{\partial\mu}\Big|_{\mu=0}+\tau(z^*)\right)\right]dF(n|\alpha)p(\alpha)=0 \tag{3}$$

where $u^*=u\left(z^*(n,\alpha)-T\left(z^*(n,\alpha)\right),\frac{z^*(n,\alpha)}{n};\alpha\right)$ and $W_u=\partial W(u;n,\alpha)/\partial u$. The tax schedule is (potentially) non-differentiable at a set of kink points $\{K_i\}$, where we denote $N_{K_i}(\alpha)$ as the set of n's for each α bunching at kink point K_i.[20] We can then split Equation 3 as follows:

$$\begin{aligned}&\sum_{\alpha\in A}\int_{N\setminus\{N_{K_i}(\alpha)\}}\left[-W_u(u^*)u_c^*\tau(z^*)+\lambda\left(T'(z^*)\frac{\partial z^*}{\partial\mu}\Big|_{\mu=0}+\tau(z^*)\right)\right]dF(n|\alpha)p(\alpha)\\&+\sum_{K_i}\sum_{\alpha\in A}\int_{N_{K_i}(\alpha)}\left[-W_u(u^*)u_c^*\tau(K_i)+\lambda\tau(K_i)\right]dF(n|\alpha)p(\alpha)=0\end{aligned} \tag{4}$$

In writing Equation 4, we use the fact that at any z for which $T(z)$ is not differentiable, almost all bunching individuals have $\frac{\partial z^*}{\partial\mu}|_{\mu=0}=0$; hence $\frac{\partial T(z^*)}{\partial\mu}|_{\mu=0}=0$.[21]

First, we deal with the term involving individuals who bunch at kinks. Let us denote the mass of individuals locating at kink K_i as $p_K(K_i)$. We then have:

$$\sum_{K_i}\sum_{\alpha\in A}\int_{N_{K_i}(\alpha)}\left[-W_u(u^*)u_c^*\tau(K_i)+\lambda\tau(K_i)\right]dF(n|\alpha)p(\alpha)=\sum_{K_i}\lambda\tau(K_i)\left(1-\overline{\omega}(K_i)\right)p_K(K_i)$$

[20] Note, we only consider kinks for which marginal tax rates are greater above the kink than below the kink (i.e., generate bunching). Kinks which feature marginal tax rates that are lower above the kink than below will always generate an unchosen set of income levels. Marginal tax rates will not be defined at unchosen income levels, so we can WLOG assume the tax schedule is differentiable at these incomes.

[21] In particular, $\frac{\partial z^*}{\partial\mu}|_{\mu=0}=0$ for all individuals locating at $z^*\in\{K_i\}$ other than the measure 0 set of individuals whose FOC holds at the limiting tax rates to the left or the right of the kink point. The remaining bunching individuals do not have their FOC satisfied: their FOC is strictly negative as we approach K_i from the left and is strictly positive as we approach K_i from the right. Thus, for sufficiently small μ, the FOC for each of these bunching individuals is still strictly negative as we approach K_i from the left and is still strictly positive as we approach K_i from the right, i.e., K_i remains their optimal income. We can arbitrarily set $\frac{\partial z^*}{\partial\mu}|_{\mu=0}=0$ for the measure 0 set of individuals whose FOC holds at the limiting tax rates as it has no impact on the integral.

where $p_K(K_i) = \sum_\alpha \int_{N_{K_i}(\alpha)} dF(n|\alpha)p(\alpha)$ and $\overline{\omega}(K_i) = \frac{1}{p_K(K_i)} \sum_\alpha \int_{N_{K_i}(\alpha)} \frac{W_u(u^*)u_c^*}{\lambda} dF(n|\alpha)p(\alpha)$ denotes the average social welfare weight at kink income K_i.

Next, we rewrite Equation 4 using the elasticity expression for $\frac{\partial z^*}{\partial \mu}|_{\mu=0}$ from Equation 2 and integrating over the optimal income distribution $H(z^*|\alpha)$ instead of the skill distribution $F(n|\alpha)$. We can do this because $z^*(n,\alpha)$ is strictly increasing in n when $T'(z^*)$ exists (see Lemma 1):

$$\sum_{\alpha \in A} \int_{Z \setminus \{K_i\}} \left[-W_u(u^*)u_c^* \tau(z^*) - \lambda \left(T'(z^*) \left(\frac{Z^c_{z^*,\alpha} z^*}{1 - T'(z^*)} \tau'(z^*) + \frac{\eta_{z^*,\alpha}}{1 - T'(z^*)} \tau(z^*) \right) - \tau(z^*) \right) \right] dH(z^*|\alpha)p(\alpha)$$
$$+ \sum_{K_i} \lambda \tau(K_i) \left(1 - \overline{\omega}(K_i)\right) p_K(K_i) = 0 \tag{5}$$

where $H(z^*|\alpha) = F(n(z^*,\alpha)|\alpha)$ and where u^*, u_c^*, Z^c, and η are now functions of (z^*, α), e.g., $u^* = u\left(z^* - T(z^*), \frac{z^*}{n(z^*,\alpha)}; \alpha\right)$.

We now consider a specific $\tau(z)$ that consists of a uniform increase in the marginal tax rate for incomes in $[\tilde{z}, \tilde{z} + d\tilde{z}]$, a uniform increase in the tax liability by $d\tilde{z}$ for incomes above $\tilde{z} + d\tilde{z} + d\tilde{z}^2$, and twice continuously differentiable expansions for incomes between $[\tilde{z} - d\tilde{z}^2, \tilde{z}]$ and for incomes between $[\tilde{z} + d\tilde{z}, z + d\tilde{z} + d\tilde{z}^2]$. In particular $\tau(z)$ equals (see Appendix A.4 for $\tau(z)$ defined over the entire domain):

$$\begin{cases} \tau(z) = 0 & \text{if } z \leq \tilde{z} - d\tilde{z}^2 \\ \tau(z) = z - \tilde{z} + d\tilde{z}^2 & \text{if } z \in [\tilde{z}, \tilde{z} + d\tilde{z}] \\ \tau(z) = d\tilde{z} & \text{if } z \geq \tilde{z} + d\tilde{z} + d\tilde{z}^2 \end{cases}$$

We choose $\tilde{z}$ and $d\tilde{z}$ such that no element of $\{K_i\}$ is in $[\tilde{z} - d\tilde{z}^2,\ \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ (this is possible by Assumption 4). Figure 2 depicts the tax schedule and the perturbed tax schedule in consumption-income space (when $\mu = 1$).[22] Note that in Figure 2, income z is a *bad* conditional on a given level of consumption so that utility is increasing to the north-west.

[22] We consider $\mu = 1$ purely for ease of graphical illustration.

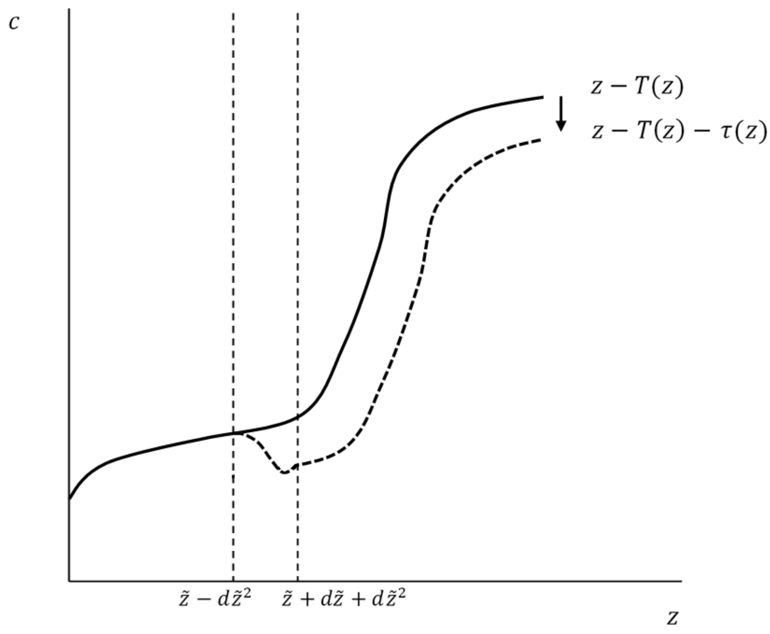

Figure 2: Perturbing the Non-Linear Tax Schedule ($\mu = 1$)

The perturbation that we consider is novel yet builds upon the approaches devised by Saez (2001), Golosov et al. (2014), and Jacquet et al. (2013). Our perturbation keeps the same valuable economic intuition from Saez's (2001) tax perturbation but is twice continuously differentiable; thus, our perturbation avoids the small bunching and jumping effects induced by kinks in Saez's (2001) perturbation. Moreover, rather than considering the effects that this perturbation has on the government's Lagrangian in a heuristic manner as in Saez (2001), we apply the insights of Golosov et al. (2014) by computing the Gateaux derivative of the government Lagrangian. Finally, our perturbation is useful in that it will allow us to explicitly analyze jumping responses that occur when individuals have multiple optimal incomes while still retaining the insight of Jacquet et al. (2013) that the implicit function theorem can be applied to ensure smooth responses to tax changes whenever individuals have unique optimal income levels.

Plugging our chosen $\tau(z)$ function into Equation 5, dividing by $d\tilde{z}\lambda$ and then taking the limit as $d\tilde{z} \to 0$, we get Proposition 1:

Proposition 1. *When all individuals have a unique optimal income level, the optimal tax schedule satisfies the following differential equation at all income levels $\tilde{z}$ for which the optimal tax schedule is differentiable:*

$$\int_{\tilde{z}}^{\infty} (1 - \bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})} \tilde{z} \bar{Z}^c_{\tilde{z}} h(\tilde{z}) - \int_{(\tilde{z},\infty)\backslash\{K_i\}} \bar{\eta}_{z^*} \frac{T'(z^*)}{1 - T'(z^*)} dH(z^*) = 0 \qquad (6)$$

where $\bar{\omega}(z^*) = \sum_\alpha \frac{W_u(u^*)u^*_c}{\lambda} p(\alpha|z^*)$ denotes the average social welfare weight at income z^*; $\bar{Z}^c_{\tilde{z}} = \sum_\alpha Z^c_{\tilde{z},\alpha} p(\alpha|\tilde{z})$ denotes the average compensated elasticity at income $\tilde{z}$; and $\bar{\eta}_{z^*} = \sum_\alpha \eta_{z^*,\alpha} p(\alpha|z^*)$ denotes the average income effect parameter at z^*.

Proof. See Appendix A.5. □

Equation 6 gives us a differential equation that the optimal tax schedule must satisfy at all points of differentiability. Equation 6 is the same as the equation derived in Saez (2001), except that in Equation 6, the elasticities and social welfare weights are averaged over the α distribution and we allow for kink points. Equivalently, Equation 6 extends the analysis from Jacquet and Lehmann (2020) to allow for kink points in the tax schedule that induce bunching (where Jacquet and Lehmann (2020) extend the analysis of Saez (2001) to the multidimensional setting when the tax schedule is everywhere differentiable and no one has multiple optima). Using the language from Saez (2001), the first term represents the mechanical effect, which captures the direct effect of our perturbation on the government's Lagrangian, holding behavioral responses of agents constant. The second term represents the elasticity effect, which captures the behavioral responses of agents earning $\tilde{z}$ to an increase in marginal tax rates at $\tilde{z}$. Finally, the third term represents the income effect, which captures the behavioral responses of agents earning above $\tilde{z}$ to an increase in their tax liability.

3.2 What If Individuals Have Multiple Optimal Incomes?

In deriving Equation 6, we assumed that all agents had one global optimal income level under the optimal tax schedule. However, there is no reason why this assumption need be true. We now derive the differential equation characterizing the optimal tax schedule allowing for a countable number of agents to have multiple optimal income levels.[23] To do so, first let us assume that there exists at most one n for each α that has multiple optimal income levels under the optimal tax schedule. Using the notation from Subsection 2.3, we denote the productivity levels of these multiple optima individuals as $m(\alpha)$.[24] Specifically, $m(\alpha)$ satisfies the following indifference condition:

$$u\left(z^{*-}(\alpha) - T(z^{*-}(\alpha)), \frac{z^{*-}(\alpha)}{m(\alpha)}; \alpha\right) = u\left(z^{*+}(\alpha) - T(z^{*+}(\alpha)), \frac{z^{*+}(\alpha)}{m(\alpha)}; \alpha\right)$$

where $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$ denote the minimum and maximum incomes chosen by type $(m(\alpha), \alpha)$, respectively. Note, we have suppressed that $z^{*-}(\alpha), z^{*+}(\alpha)$, and $m(\alpha)$ are also functions of the tax schedule. Using this notation, we rewrite the government Lagrangian as follows:

$$\begin{aligned}\mathcal{L} = &\sum_{\alpha \in A} \int_0^{m(\alpha)} \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T\left(z^*(n,\alpha)\right)\right] dF(n|\alpha)p(\alpha) + \\ &\sum_{\alpha \in A} \int_{m(\alpha)}^{\infty} \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T\left(z^*(n,\alpha)\right)\right] dF(n|\alpha)p(\alpha)\end{aligned}$$

Before deriving how the government's Lagrangian changes in response to a tax perturbation, we first explore how types with multiple optima change in response to a tax perturbation. We

[23] Lemma 2 ensures the number of individuals with multiple optimal incomes is countable.

[24] If no type α individual has multiple optimal incomes, set $m(\alpha) = 0$.

do this by differentiating the indifference condition with respect to μ:

$$\frac{\partial}{\partial \mu}\left[u\left(z^{*-}(\alpha)-T(z^{*-}(\alpha))-\mu\tau(z^{*-}(\alpha)),\frac{z^{*-}(\alpha)}{m(\alpha)};\alpha\right)-\right.$$
$$\left.\left.u\left(z^{*+}(\alpha)-T(z^{*+}(\alpha))-\mu\tau(z^{*+}(\alpha)),\frac{z^{*+}(\alpha)}{m(\alpha)};\alpha\right)\right]\right|_{\mu=0}=0 \tag{7}$$

noting that $m(\alpha), z^{*-}(\alpha)$, and $z^{*+}(\alpha)$ are all functions of μ. However, by the agent's first order condition, the derivatives of the first term and second term of Equation 7 w.r.t. $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$ (respectively) are zero; thus, both $\partial z^{*-}(\alpha)/\partial\mu$ and $\partial z^{*+}(\alpha)/\partial\mu$ are multiplied by zero. Rearranging for $\partial m(\alpha)/\partial\mu|_{\mu=0}$, we get:[25]

$$\left.\frac{\partial m(\alpha)}{\partial \mu}\right|_{\mu=0}=\frac{u_c^{*+}(\alpha)\,\tau\left(z^{*+}(\alpha)\right)-u_c^{*-}(\alpha)\,\tau\left(z^{*-}(\alpha)\right)}{u_l^{*-}(\alpha)\,\frac{z^{*-}(\alpha)}{m(\alpha)^2}-u_l^{*+}(\alpha)\,\frac{z^{*+}(\alpha)}{m(\alpha)^2}} \tag{8}$$

where $u_c^{*+}(\alpha)=u_c\left(z^{*+}(\alpha)-T(z^{*+}(\alpha)),\frac{z^{*+}(\alpha)}{m(\alpha)};\alpha\right)$ and $u_l^{*-}(\alpha)=u_l\left(z^{*-}(\alpha)-T(z^{*-}(\alpha)),\frac{z^{*-}(\alpha)}{m(\alpha)};\alpha\right)$ etc.

Next, we use Leibniz's integral rule to take the derivative of the government Lagrangian in the direction of $\tau(z)$, starting from the optimal tax schedule:

$$\sum_{\alpha\in A}\int_0^\infty\left[-W_u(u^*)u_c^*\tau(z^*)+\lambda\left(\left.\frac{\partial T(z^*)}{\partial\mu}\right|_{\mu=0}+\tau(z^*)\right)\right]dF(n|\alpha)p(\alpha)+$$
$$\sum_{\alpha\in A}\left[W\left(u^{*-}(\alpha)\right)+\lambda T(z^{*-}(\alpha))\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}\right]f(m(\alpha)|\alpha)p(\alpha)-$$
$$\sum_{\alpha\in A}\left[W\left(u^{*+}(\alpha)\right)+\lambda T(z^{*+}(\alpha))\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}\right]f(m(\alpha)|\alpha)p(\alpha)=0$$

Note that in the first term above, we integrate over the entire set of individuals (even those with multiple optima). The value that we assign to $\frac{\partial T(z^*)}{\partial\mu}|_{\mu=0}$ for those with multiple optima is irrelevant because the set with multiple optima is countable and hence measure 0, so does not affect the integral in the first term. Noting that $u^{*-}(\alpha)=u^{*+}(\alpha)\ \forall\ \alpha$, we get:

$$\sum_{\alpha\in A}\int_0^\infty\left[-W_u(u^*)u_c^*\tau(z^*)+\lambda\left(\left.\frac{\partial T(z^*)}{\partial\mu}\right|_{\mu=0}+\tau(z^*)\right)\right]dF(n|\alpha)p(\alpha)+$$
$$\underbrace{\sum_{\alpha\in A}\lambda\left(T\left(z^{*-}(\alpha)\right)-T\left(z^{*+}(\alpha)\right)\right)\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}f(m(\alpha)|\alpha)}_{\text{jumping effects}}p(\alpha)=0 \tag{9}$$

Equation 9 is the same as Equation 3 except Equation 9 includes jumping effects. These jumping effects capture the fact that when we perturb the tax schedule, each type $(m(\alpha),\alpha)$ may now strictly prefer one of their optimal income levels and thus jump to the income level

[25] Even if $T(z)$ is not differentiable at z^{*-} (and/or z^{*+}), Equation 8 still holds. See Appendix A.6.

they prefer.[26] Moreover, if type $(m(\alpha), \alpha)$ jumps for any given perturbation to the tax schedule, a small mass of individuals with type α and n close to $m(\alpha)$ will also jump in response to the change in the tax schedule (as optimal utility is continuous in n for each α).

We now explore the jumping effects in Equation 9 in more detail. We know by Equation 8 that the value of these effects will depend on the tax changes experienced at $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$. Let's consider the same specific $\tau(z)$ function we did earlier:[27]

$$\begin{cases} \tau(z) = 0 & \text{if } z \leq \tilde{z} - d\tilde{z}^2 \\ \tau(z) = z - \tilde{z} + d\tilde{z}^2 & \text{if } z \in [\tilde{z}, \tilde{z} + d\tilde{z}] \\ \tau(z) = d\tilde{z} & \text{if } z \geq \tilde{z} + d\tilde{z} + d\tilde{z}^2 \end{cases}$$

where we choose $\tilde{z}$ and $d\tilde{z}$ such that no element of $\{z_i^{mult}\}$ or $\{K_i\}$ is within $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ (this is possible by Assumption 3 and Assumption 4).

First, consider an α with $z^{*+}(\alpha) < \tilde{z} - d\tilde{z}^2$. Because $\tau(z^{*+}(\alpha)) = \tau(z^{*-}(\alpha)) = 0$, we have $\partial m(\alpha)/\partial\mu|_{\mu=0} = 0$. Thus, the jumping effect for this α is equal to 0. Next consider an α with $z^{*-}(\alpha) < \tilde{z} - d\tilde{z}^2$ and $z^{*+}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$. For this α, $\tau(z^{*+}(\alpha)) = d\tilde{z}$ and $\tau(z^{*-}(\alpha)) = 0$ so that the jumping effect in Equation 9 is equal to:

$$\lambda\left(T\left(z^{*-}(\alpha)\right) - T\left(z^{*+}(\alpha)\right)\right)\frac{u_c^{*+}(\alpha)\, d\tilde{z}}{u_l^{*-}(\alpha)\frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha)\frac{z^{*+}(\alpha)}{m(\alpha)^2}} f(m(\alpha)|\alpha) \equiv J_1(\alpha)\lambda d\tilde{z}$$

Figure 3 depicts the jumping behavioral response captured by $J_1(\alpha)$. Specifically, Figure 3 shows an individual $(m(\alpha), \alpha)$ with two optimal income levels, $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$, under the optimal tax schedule, where $z^{*+}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$ and $z^{*-}(\alpha) < \tilde{z} - d\tilde{z}^2$. Under the perturbed schedule, our indifferent individual now strictly prefers $z^{*-}(\alpha)$ to $z^{*+}(\alpha)$ given she experiences a drop in consumption of $d\tilde{z}$ at $z^{*+}(\alpha)$ but not at $z^{*-}(\alpha)$. Consequently, $(m(\alpha), \alpha)$ will jump down to $z^{*-}(\alpha)$. This jumping movement is captured by the red arrow in Figure 3 below.

Now consider an α with $z^{*-}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$. For this α, $\tau(z^{*+}(\alpha)) = \tau(z^{*-}(\alpha)) = d\tilde{z}$ so that the jumping effect in Equation 9 is equal to:

$$\lambda\left(T\left(z^{*-}(\alpha)\right) - T\left(z^{*+}(\alpha)\right)\right)\frac{u_c^{*+}(\alpha)\, d\tilde{z} - u_c^{*-}(\alpha)\, d\tilde{z}}{u_l^{*-}(\alpha)\frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha)\frac{z^{*+}(\alpha)}{m(\alpha)^2}} f(m(\alpha)|\alpha) \equiv J_2(\alpha)\lambda d\tilde{z}$$

Figure 4 depicts the jumping behavioral response captured by $J_2(\alpha)$. Specifically, Figure 4 shows an individual $(m(\alpha), \alpha)$ with two optimal income levels, $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$, under the optimal tax schedule, where $z^{*-}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$. Under the perturbed schedule, our indifferent

[26] How do we know that $(m(\alpha), \alpha)$ will jump from his minimum optimal income $z^{*-}(\alpha)$ to his maximum optimal income $z^{*+}(\alpha)$ or vice versa (as opposed to jumping to another income between these two levels)? After the perturbation, there will be a new individual of type α with multiple optima, where, by continuity, his minimum and maximum optimal incomes will be very close to $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$. By the (SCP), no other individual of type α can locate in between these two new optimal incomes, hence, $(m(\alpha), \alpha)$ must have jumped to an income right around $z^{*+}(\alpha)$ or $z^{*-}(\alpha)$.

[27] Again, see Appendix A.4 for a definition of $\tau(z)$ over its entire domain.

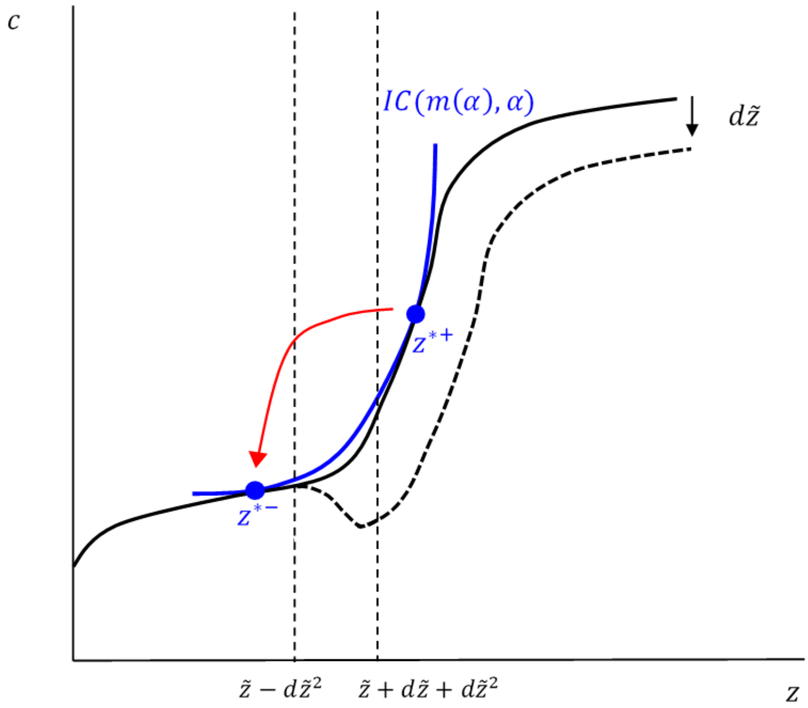

Figure 3: $J_1(\alpha)$ Jumping Effect ($\mu = 1$)

individual will typically prefer $z^{*+}(\alpha)$ to $z^{*-}(\alpha)$ given she experiences a drop in consumption of $d\tilde{z}$ at both income levels which is more costly (in terms of a reducing utility) at $z^{*-}(\alpha)$ due to concavity of utility. Consequently, $(m(\alpha), \alpha)$ will typically jump up to a new optimal income level very close to $z^{*+}(\alpha)$. This jumping movement is captured by the red arrow in Figure 4 below. Note, however, if preferences are quasi-linear in consumption, this individual will not jump as $u_c^{*+}(\alpha) = u_c^{*-}(\alpha)$, implying $J_2(\alpha) = 0$.[28]

Because we choose $\tilde{z}$ and $d\tilde{z}$ s.t. $\{z_i^{mult}\} \notin [\tilde{z} - d\tilde{z}^2,\, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$, we do not need to consider the possibility that either $z^{*-}(\alpha)$ or $z^{*+}(\alpha)$ lie inside $[\tilde{z} - d\tilde{z}^2,\, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$. Thus, there are no other jumping effects that can occur as a result of our tax perturbation. Dividing Equation 9 by $d\tilde{z}\lambda$ and taking the limit as $d\tilde{z} \to 0$ as in Subsection 3.1, we get:

$$\begin{aligned}&\int_{\tilde{z}}^{\infty} (1 - \bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})} \tilde{z} \bar{Z}^c_{\tilde{z}} h(\tilde{z}) - \int_{(\tilde{z},\infty)\setminus\{K_i\}} \bar{\eta}_{z^*} \frac{T'(z^*)}{1 - T'(z^*)} dH(z^*) \\ &+ \sum_{\alpha \in A} \left[J_1(\alpha) \mathbb{1}\left(z^{*-}(\alpha) < \tilde{z} < z^{*+}(\alpha)\right) + J_2(\alpha) \mathbb{1}\left(z^{*-}(\alpha) > \tilde{z}\right)\right] p(\alpha) = 0\end{aligned} \tag{10}$$

Equation 10 gives us a condition that the optimal tax schedule must satisfy at all $\tilde{z} \notin \{z_i^{mult}\} \cup \{K_i\}$ under the assumption that there exists at most one n for each α with multiple optimal income levels.

Finally, we relax the assumption that there exists at most one n for each α with multiple optima. Instead, we allow there to exist a countable number of $n's$ with multiple optima for

[28] Alternatively, if u_{cl} is sufficiently positive (so that consumption and leisure are strong substitutes), this individual will jump down as she now prefers $z^{*-}(\alpha)$ to $z^{*+}(\alpha)$.

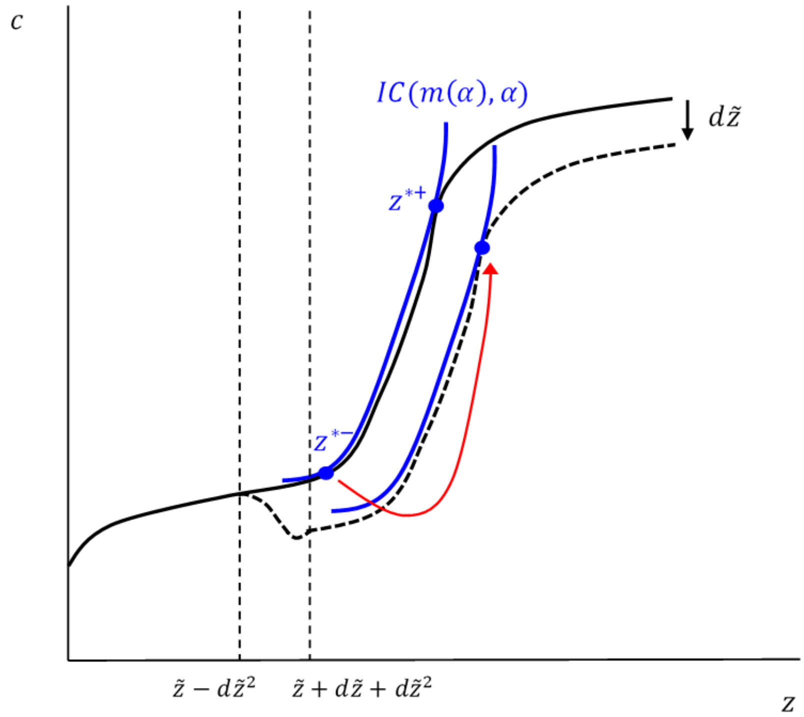

Figure 4: $J_2(\alpha)$ Jumping Effect ($\mu = 1$)

each α. Using the notation from Subsection 2.3, we denote $m_i(\alpha)$ as the i^{th} productivity level with multiple optimal incomes, and denote their minimum optimal income as $z_i^{*-}(\alpha)$ and their maximum optimal income as $z_i^{*+}(\alpha)$. Finally, we denote the number of individuals with multiple incomes for a given α as $M(\alpha)$ (which can also be countably infinite or zero). We can augment Equation 10 to yield a general formula for the optimal tax schedule that allows for the possibility that some individuals have multiple optimal income levels:

Proposition 2. *The optimal tax schedule satisfies the following differential equation at all income levels $\tilde{z} \notin \{z_i^{mult}\}$ at which the optimal tax schedule is differentiable:*

$$\int_{\tilde{z}}^{\infty} (1-\bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}\bar{Z}_{\tilde{z}}^{c} h(\tilde{z}) - \int_{(\tilde{z},\infty)\setminus\{K_i\}} \bar{\eta}_{z^*}\frac{T'(z^*)}{1-T'(z^*)}dH(z^*)$$
$$\sum_{\alpha\in A}\sum_{i=1}^{M(\alpha)} \left[J_{1i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha)<\tilde{z}<z_i^{*+}(\alpha)\right) + J_{2i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha)>\tilde{z}\right)\right]p(\alpha) = 0 \tag{11}$$

Proof. See Appendix A.7. □

Equation 11 extends the optimality condition derived in Jacquet and Lehmann (2020) to allow for the possibility that individuals have multiple optimal income levels (and to allow for bunching at kink points).[29] The possibility of multiple optima individuals results in the inclusion of the jumping effect terms, J_1 and J_2. $J_{1i}(\alpha)$ is negative provided that $T\left(z_i^{*+}(\alpha)\right) -$

[29]When heterogeneity is unidimensional, Equation 11 is equivalent to the optimality condition from Mirrlees (1971); see Appendix A.8.

$T\left(z_i^{*-}(\alpha)\right) > 0$.[30] Hence, ignoring J_1 jumping effects will typically lead us to *overestimate* the welfare impact of a small tax increase. However, $J_{2i}(\alpha)$ is (weakly) positive provided that $T\left(z_i^{*+}(\alpha)\right) - T\left(z_i^{*-}(\alpha)\right) > 0$ and that consumption and leisure are not sufficiently strong substitutes. Hence, ignoring J_2 jumping effects will typically lead us to (weakly) *underestimate* the welfare impact of a small tax increase.

4 Theoretical Results about Individuals with Multiple Optima

Up to this point, we have derived an equation for the optimal tax schedule with multidimensional agent heterogeneity, explicitly accounting for the possibility that individuals have multiple optimal income levels. In this section, we provide a partial characterization of when the optimal tax schedule will exhibit individuals with multiple optima.

4.1 No Individuals with Multiple Optima

When employing the tax perturbation approach, papers in this literature have typically assumed that all individuals respond smoothly to tax changes, thus indirectly assuming that no individual has multiple optimal income levels. While it turns out to be impossible to rule out the existence of individuals with multiple optima when we have multiple dimensions of heterogeneity, with *one dimension of heterogeneity*, we can rule out individuals having multiple optimal income levels. Proposition 3 gives conditions on primitives such that no individuals will have multiple optima under the optimal tax schedule.

Proposition 3. *If individuals only differ in terms of productivity (so that everyone has the same value of* α*), indifference curves are convex in* (c, z) *space, and* $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ *is increasing in* z *along each individual's indifference curve, then no individual has multiple optimal income levels under the optimal tax schedule; hence, no jumping behavioral responses are present under the optimal tax schedule.*[31]

Proof. See Appendix A.10 □

Proposition 3 is inspired by Theorem 2(v) in Mirrlees (1971), although Mirrlees states the result without proof. Mirrlees states that in order to rule out individuals with multiple optima, $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ must be increasing in z along each individual's indifference curve. Mirrlees does not state that the productivity density need be continuous nor does he place a restriction

[30] Positive marginal tax rates imply $T\left(z_i^{*+}(\alpha)\right) - T\left(z_i^{*-}(\alpha)\right) > 0$. See Appendix A.9 for a proof that $J_{1i}(\alpha) < 0$ assuming $T\left(z_i^{*+}(\alpha)\right) - T\left(z_i^{*-}(\alpha)\right) > 0$.

[31] Where $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ increasing in z along each individual's indifference curve means that $\frac{\partial}{\partial z}\left(\frac{z}{n}u_{cl}\left(\hat{c}, \frac{z}{n}\right)\frac{u_l\left(\hat{c}, \frac{z}{n}\right)}{u_c\left(\hat{c}, \frac{z}{n}\right)} - u_l\left(\hat{c}, \frac{z}{n}\right) - \frac{z}{n}u_{ll}\left(\hat{c}, \frac{z}{n}\right)\right) > 0 \;\forall\; n, \bar{u}$, where $\hat{c}(z; n, \bar{u})$ solves $u\left(\hat{c}, \frac{z}{n}\right) = \bar{u}$.

on the shape of indifference curves; however, we believe both of these restrictions are necessary. Proposition 2.6 from Hellwig (2010), which also formally proves Theorem 2(v) from Mirrlees (1971), requires that the productivity density is continuous, $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ is increasing in z along each individual's indifference curve, and that an additional condition on utility is satisfied, which, in turn, implies convexity of indifference curves. We believe our proof (which is entirely different from the proof of Proposition 2.6 in Hellwig (2010)) is helpful towards understanding the underlying intuition behind why the conditions in Proposition 3 imply that jumping effects cannot occur under the optimal tax schedule.

A sketch of the proof for Proposition 3 goes as follows: suppose there is an individual with two optima under the optimal tax schedule. Consider a perturbation that increases the marginal tax rate at an income just below z^{*-} but keeps the individual with multiple optima indifferent between her two optima. This is possible to do: we simply change tax rates in the region (z^{*-}, z^{*+}) so that $\tau(z^{*+}) = \frac{u_c^{*-}}{u_c^{*+}}\tau(z^{*-})$ (see Equation 8 above). Note, changing tax rates in the region (z^{*-}, z^{*+}) does not create any behavioral responses in this region because, by (SCP), no one locates in (z^{*-}, z^{*+}) as we only have one dimension of heterogeneity. Such a perturbation results in an elasticity effect for the individual with income just less that z^{*-}, and an income and mechanical effect for all those with incomes $z^* > z^{*+}$. Jumping effects are 0 under this perturbation given we specifically chose a perturbation that keeps the individual with multiple optima unchanged. Now consider increasing the marginal tax rate at an income just above z^{*+}. This induces an elasticity effect for the individual just above z^{*+}, and income and mechanical effects for all those with incomes $z^* > z^{*+}$. Under the optimal tax schedule, the effect of both perturbations should be 0. However, our assumptions on the shape of indifference curves in Proposition 3 give us that the elasticity effect at z^{*+} is smaller than the elasticity effect at z^{*-}. Thus, given both perturbations lead to the same income and mechanical effects but the former perturbation induces a larger elasticity effect, the total effect of both perturbations cannot both be 0. Hence, we could not have been at the optimal tax schedule.

Note, Proposition 3 does *not* give a condition on preferences such that no one will have multiple optima for any tax schedule - this is impossible under reasonable assumptions (for example, a simple indifference curve argument shows that an individual will have multiple optima if the marginal tax rate schedule is piece-wise linear with decreasing rates). As such, an understanding of jumping behavioral responses will be necessary *a priori* when analyzing the effects of tax reforms starting from sub-optimal tax schedules.

Next, it is worthwhile to discuss the assumptions in Proposition 3. First, we remind the reader that a maintained assumption throughout this paper is that $F(n|\alpha)$ is continuously differentiable $\forall\alpha$ (so that $F(n)$ is continuous if all individuals have the same α). This assumption is important because Hellwig (2010) shows that an individual will have multiple optima in the case of unidimensional heterogeneity whenever the productivity distribution has mass points. Second, it seems reasonable to assume that individuals' indifference curves are convex in (c, z) space - this is a standard assumption on preferences requiring that individuals be compensated with more and more consumption as their labor supply increases. The other assumption in Proposition 3, $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ is increasing in z for fixed utility and n, seems relatively obtuse and it is not readily clear that this assumption is sensible. However, Corollary 1 shows that, in fact, this assumption applies in a wide class of common utility functions:

Corollary 1. *If utility takes the form:* $u(c, z; n) = v(c) - (z/n)^{1+k}/(1 + k)$ *where* $k > 0$*, no individual will have multiple optima under the optimal schedule.*

Proof. This follows from the conditions in Proposition 3. □

Moreover, through trial and error it seems exceedingly difficult to find a utility function that satisfies standard properties $u_c > 0, u_l < 0, u_{cc} \leq 0, u_{ll} < 0$ as well as the (SCP) yet violates the condition that $\frac{z}{n} u_{cl} \frac{u_l}{u_c} - u_l - \frac{z}{n} u_{ll}$ is increasing in z along each individual's indifference curve.

Finally, it is worth highlighting a special class of utility functions with multiple dimensions of heterogeneity for which no individual will have multiple optima under the optimal schedule. The idea behind this result is that some special classes of utility functions with two dimensions of heterogeneity yield an optimal tax problem that is isomorphic to a one dimensional problem so that we can apply Proposition 3.

Corollary 2. *Suppose utility takes the form:* $u(c, z/n; \alpha) = \alpha v(c) - (z/n)^{1+k}/(1 + k)$*. If* $k > 0$ *and* $F(n)$ *is continuously differentiable, all individuals will have a unique optimum under the optimal tax schedule.*[32]

Proof. See Appendix A.11. □

4.2 Individuals with Multiple Optima

If there is only one dimension of heterogeneity, we can generally rule out individuals with multiple optimal income levels under the optimal tax schedule. However, it turns out to be impossible to extend this result to multiple dimensions. We now proceed to show that it is *not* possible to find a multidimensional analogue to Proposition 3 in cases with arbitrary multidimensional heterogeneity. In particular, we show that it is possible for individuals to have multiple optima under the optimal schedule even if the second dimension of heterogeneity is binary (so that there are only two values of α), the productivity distributions $F(n|\alpha)$ are continuously differentiable $\forall \alpha$, and the utility functions of each type satisfy standard assumptions:[33]

Proposition 4. *Suppose there are two* α *types, each with a different utility function. There exist two utility functions that satisfy (SCP) and the conditions in Proposition 3 yet lead to an*

[32] Throughout this paper, we assume that $F(n|\alpha)$ is continuously differentiable $\forall \alpha$. To prove Corollary 2, we require that $F(n)$ is continuously differentiable. $F(n)$ will be continuously differentiable if either the support of $F(n|a)$ is the same $\forall \alpha$ or if $f(n|\alpha) \to 0$ $\forall \alpha$ at the endpoints of their support.

[33] Proposition 2.4 from Hellwig (2010) proves that some individual will have multiple optimal income levels in a unidimensional tax model whenever the productivity density has mass points. Conversely, Proposition 4 shows that the optimal schedule can lead to an individual with multiple optima even when $F(n|\alpha)$ is continuously differentiable $\forall \alpha$.

individual with multiple optima under the optimal tax schedule (i.e., jumping effects are present under the optimal tax schedule).

Proof. See Appendix A.12. □

The underlying intuition for Proposition 4 is as follows. As seen in Figure 3, in order for an individual to have multiple optima we need (a) marginal tax rates to decrease over a certain portion of the income distribution (so that a portion of consumption schedule is convex), and (b) indifference curves for at least one type to not be overly steep. Suppose that there are two types of individuals, $\{\alpha_1, \alpha_2\}$, each with a different utility function: $u(c, z/n; \alpha_i) = u^{(i)}(c, z/n)$ for $i = \{1, 2\}$. Further suppose that, for a given productivity level n, α_2 individuals have steeper indifference curves compared to α_1 individuals. Now, consider a population that consists only of α_2 individuals so that there is only productivity heterogeneity. Mirrlees (1971) showed that when individuals only differ in terms of how productive they are, tax rates will always be between 0 and 1. Moreover, tax rates must always be 0 at the top and the bottom of the income distribution (see Propositions 6 and 7 below); hence, when all individuals are type α_2, there will be a region of decreasing tax rates in the optimal tax schedule. Thus, provided that the utility function we choose for α_1 generates sufficiently flat indifference curves, we will have at least one α_1 individual that has multiple optimal income levels under the optimal tax schedule that ensues when only type α_2 individuals exist. Finally, consider a population that consists of both α_1 and α_2 individuals. We can use continuity arguments to show that as the proportion of α_2 individuals in society becomes arbitrarily close to 1, the optimal tax schedule gets arbitrarily close to the optimal tax schedule that ensues when only type α_2 individuals exist; hence, at least one α_1 individual will have multiple optimal income levels when society consists of some arbitrarily small, yet positive, proportion of type α_1 individuals.

Finally, we show that the optimal tax schedule will be non-differentiable at any income level for which someone has multiple optimal incomes as long as the optimal tax schedule, $T(z)$, is everywhere increasing:

Proposition 5. *If indifference curves are convex in (c, z) space, $\frac{z}{n} u_{cl} \frac{u_l}{u_c} - u_l - \frac{z}{n} u_{ll}$ is increasing in z along each individual's indifference curve, and $T(z)$ is everywhere increasing, optimal marginal tax rates increase discontinuously (generating bunching) $\forall\ z \in \{z_i^{mult}\}$.*

Proof. See Appendix A.13. □

Proposition 5 is important for performing optimal tax simulations: any time an individual has multiple optimal incomes, there will typically be a kink point in the optimal tax schedule that induces bunching.

4.3 Additional Results

Finally, we briefly present two additional results that will be useful for conducting simulations:

Proposition 6. *Optimal marginal tax rates are 0 at the bottom of the income distribution, $T'(\underline{z}) = 0$, as long as the support of the skill distribution, supp($f(n|\alpha)$), is closed and bounded $\forall$ α, and $\underline{z} \notin \{z_i^{mult}\}$ where $\underline{z}$ denotes the lowest income chosen in society under the optimal tax schedule.*

Proof. See Appendix A.14. □

Proposition 7. *Optimal marginal tax rates are 0 at the top of the income distribution, $T'(\bar{z}) = 0$, as long as the support of the skill distribution, supp($f(n|\alpha)$), is closed and bounded $\forall$ α, and $\bar{z} \notin \{z_i^{mult}\}$ where $\bar{z}$ denotes the highest income chosen in society under the optimal tax schedule.*

Proof. See Appendix A.15. □

Propositions 6 and 7 extend the classic results of Sadka (1976) and Seade (1977) to the case of multidimensional agent heterogeneity. While Diamond (1998), Saez (2001), and Diamond and Saez (2011) have argued that the "zero marginal rates at the top and bottom" are highly local and therefore likely policy irrelevant, these results are nonetheless helpful for conducting numerical simulations as they provide boundary conditions for Equation 11.

5 Optimal Tax Simulations

This section derives a method that can be used to simulate optimal income tax schedules with multidimensional agent heterogeneity and illustrates how to apply this method via a numerical example. There are two goals of this numerical exercise: (1) to show that our novel simulation method yields identical results to Mirrlees's Hamiltonian simulation method when heterogeneity is unidimensional, and (2) to illustrate that our method can solve for tax schedules when agent heterogeneity is multidimensional and is able to handle the possibility that some individuals have multiple optimal income levels.

5.1 General Simulation Methodology

Recall that Equation 11 characterizes the optimal tax schedule at all income levels $\tilde{z}$ where the marginal tax rate exists and all individuals at $\tilde{z}$ have no other optimal incomes. However, even ignoring jumping effects, this differential equation is difficult to solve directly as the optimal marginal tax rate at $\tilde{z}$ depends on the tax schedule everywhere above $\tilde{z}$ through the income and mechanical effects. The presence of jumping effects amplifies the difficulty of solving Equation 11 as quantifying jumping effects requires knowing which individuals have multiple optimal income levels and what their multiple optimal income levels are (which in turn requires knowing the entire tax schedule). Previous optimal tax simulations have thus either (1) used Mirrlees's Hamiltonian approach in unidimensional applications even if they derive the schedule in terms of observable sufficient statistics (e.g., Saez (2001) or Lockwood and Weinzierl (2016)) or (2) assumed away jumping effects and iterated on Equation 11 until convergence (e.g., Mankiw et al. (2009) or Jacquet and Lehmann (2020)). However, the Hamiltonian approach does not appear to be feasible with multidimensional heterogeneity given the number of incentive compatibility constraints, and we found the iterative method impossible to adapt to the situation in which some individuals have multiple optimal income levels due to numerical instability.[34]

As such, we derive a new method to simulate optimal income tax schedules. First, note that Equation 11 holds at all income levels $\tilde{z}$ except for a countable set of income levels (which are separated by a minimum distance) at which multiple optima individuals locate and at income levels where the tax schedule is non-differentiable. As such, the derivative of Equation 11 with respect to $\tilde{z}$ must also be 0 on all the intervals which do not contain an individual with multiple optima or a non-differentiable point of the tax schedule. Differentiating Equation 11 is useful because the jumping effects are not a function of $\tilde{z}$, so that the derivative of the jumping terms with respect to $\tilde{z}$ are zero. Differentiating Equation 11 with respect to $\tilde{z}$ we arrive at:

$$\sum_{\alpha \in A}\left[-1+\omega(\tilde{z},\alpha)+\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\eta_{\tilde{z},\alpha}\right]h(\tilde{z}|\alpha)p(\alpha)-\frac{\partial}{\partial \tilde{z}}\left[\sum_{\alpha \in A}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}Z^c_{\tilde{z},\alpha}h(\tilde{z}|\alpha)p(\alpha)\right]=0 \tag{12}$$

where $\omega(\tilde{z},\alpha)=\frac{W_u(\tilde{u})\tilde{u}_c}{\lambda}$ denotes the social welfare weight on individual $(n(\tilde{z},\alpha),\alpha)$ with $\tilde{u}=u\left(\tilde{z}-T(\tilde{z}),\frac{\tilde{z}}{n(\tilde{z},\alpha)};\alpha\right)$.

Equation 12 is a third order differential equation (the compensated elasticity Z^c depends on the second derivative $T''(z)$, hence the derivative of Z^c depends on the third derivative $T'''(z)$). As with Equation 11, Equation 12 holds for all $\tilde{z}$ at which the tax schedule is smooth and no individual locating there has a second optimal income. We can simplify Equation 12 into a second order differential equation by substituting in $h(\tilde{z}|\alpha)=f(n(\tilde{z},\alpha)|\alpha)\frac{\partial n(z^*,\alpha)}{\partial z^*}|_{z^*=\tilde{z}}$ along with the full expressions for $\frac{\partial n(z^*,\alpha)}{\partial z^*}|_{z^*=\tilde{z}}$ and $Z^c_{\tilde{z},\alpha}$ (see Appendix A.16 for the derivation of Equation 13):

$$\begin{aligned}&\sum_{\alpha \in A}\left[-1+\omega(\tilde{z},\alpha)+\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\eta_{\tilde{z},\alpha}\right]\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}}f(n(\tilde{z},\alpha)|\alpha)p(\alpha)+\\&\frac{\partial}{\partial \tilde{z}}\left[\sum_{\alpha \in A}\frac{\tilde{u}_cT'(\tilde{z})}{\frac{\tilde{z}}{n^2}\tilde{u}_{cl}(1-T'(\tilde{z}))+\frac{1}{n^2}\tilde{u}_l+\frac{\tilde{z}}{n^3}\tilde{u}_{ll}}f(n(\tilde{z},\alpha)|\alpha)p(\alpha)\right]=0\end{aligned} \tag{13}$$

where $\tilde{u}_c=u_c\left(\tilde{z}-T(\tilde{z}),\frac{\tilde{z}}{n(\tilde{z},\alpha)};\alpha\right)$, etc.

Let's now discuss how one could use Equation 13 to solve for optimal tax rates. We will assume that the tax schedule is always twice continuously differentiable except at points where individuals have multiple optimal incomes (note, by Proposition 5, we cannot assume that the tax schedule is differentiable at these points).[35] First, suppose that no individual has

[34] We found that small changes in the tax schedule which cause individuals to jump were not handled well numerically.

[35] Any tax schedule we find to be optimal within the class of functions that are smooth except at

multiple optimal income levels so that the optimal tax schedule is everywhere twice continuously differentiable (this is an endogenous assumption and needs to be verified once we have solved for the optimal tax schedule). We can then solve Equation 13 using standard differential equation techniques given initial values for $T'(\underline{z})$ and $T(\underline{z})$ as well as the Lagrange multiplier, λ. As we show in Proposition 6, with a closed and bounded skill distribution, the marginal tax rate on the lowest income chosen in society, $T'(\underline{z})$, is 0. Given the initial condition $T'(\underline{z}) = 0$, our simulation procedure then searches over the values of $T(\underline{z})$ and λ that maximize total social welfare subject to the government's budget constraint.[36] Finally, we check to ensure that the income level assigned to each individual is, in fact, their unique global optimal income, i.e., we check that $z_{FOC}(n, \alpha) = z^*(n, \alpha)$ where $z_{FOC}(n, \alpha)$ denotes the income schedule that results from solving $(n, \alpha)'s$ FOC under the proposed optimal tax schedule.

If our above method yields an income schedule z_{FOC} s.t. $z_{FOC}(n, \alpha) \neq z^*(n, \alpha)$ for some (n, α), then our proposed optimal tax schedule is not valid given it does not satisfy agent's incentive compatibility constraints. We therefore augment our procedure to allow for individuals to have multiple optima. First, we assume that only one individual, (m, a), has two optimal income levels under the optimal tax schedule. Now, we must search over the parameter space of $\lambda, T(\underline{z}), (m, a), T'_{j1}, T'_{j2}$ to find the values that maximize total welfare subject to the government's budget constraint, where T'_{j1}, T'_{j2} denote the jumps in the marginal tax rates that occur at $(m, a)'s$ minimum and maximum optimal incomes.[37] We then check to ensure that the income level assigned to each individual is, in fact, their unique global optimal income given the solution to Equation 13, other than for individual (m, a), who has two global optimal incomes. If not, we repeat the above exercise but assume there exist two individuals with multiple optima, etc.

5.2 Specific Numerical Example

We now discuss a specific numerical example with multidimensional heterogeneity. The second dimension of heterogeneity α captures the curvature of the disutility over labor, which is inversely proportional to the elasticity of earnings with respect to the tax rate. First, we assume the second dimension of heterogeneity is binary (i.e., α takes on two values), the government

$\{z_i^{mult}\}$ will actually be optimal in the broader class of all piece-wise smooth functions. The logic is the same as the logic behind the result that any function that is optimal within the class of twice continuously differentiable functions is optimal within the class of all piece-wise smooth functions (e.g., see Roughan (2016)).

[36] Note, as is always the case when simulating optimal tax schedules, we are finding a locally optimal tax schedule, not necessarily a globally optimal tax schedule.

[37] Roughly speaking, for a given set of parameter values, we use Equation 13 to solve for tax rates until we hit the income level s.t. $(m, a)'s$ FOC is satisfied. We call this income level z^{*-} and add T'_{j1} to the marginal tax rate. We then set the income density for type a to 0 as we know by the (SCP) that no a type can locate between (z^{*-}, z^{*+}). We then continue to use Equation 13 to solve for tax rates until we get to an income level, denoted z^{*+}, where utility at z^{*+} is equal to utility at z^{*-} for type (m, a). We then turn the density for type a back on and add T'_{j2} to the marginal tax rate at z^{*+}. Note this is only a rough outline - for exact details (e.g., how we deal with bunching at the kink points), see Appendix C.1.

social welfare function is defined by $W(u;n,\alpha)=\log(u)$, and utility takes the following form:[38]

$$u\left(z-T(z),\frac{z}{n};\alpha_i\right)=z-T(z)-\frac{\left(\frac{z}{n}\right)^{1+\alpha_i}}{1+\alpha_i}\text{ for } i=1,2$$

where $\alpha_1=2,\alpha_2=4$. Note, if marginal tax rates are constant, compensated elasticities are inversely proportional to α_i, i.e., $Z^c_{\alpha_1}=\frac{1}{2},Z^c_{\alpha_2}=\frac{1}{4}$. With this utility function and government social welfare function, Equation 13 becomes:

$$\begin{aligned}&\sum_{i=1}^{2}\left(1-\omega(\tilde{z},\alpha_i)\right)\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}}f(n(\tilde{z},\alpha_i)|\alpha_i)p(\alpha_i)+\\&\frac{\partial}{\partial\tilde{z}}\left[\sum_{i=1}^{2}\frac{T'(\tilde{z})}{\frac{1}{n^2}(1+\alpha_i)\left(\frac{\tilde{z}}{n}\right)^{\alpha_i}}f(n(\tilde{z},\alpha_i)|\alpha_i)p(\alpha_i)\right]=0\end{aligned}\tag{14}$$

where $\omega(\tilde{z},\alpha_i)=\frac{1}{\lambda u(\tilde{z},\alpha_i)}$, $u(\tilde{z},\alpha_i)=\tilde{z}-T(\tilde{z})-\frac{\left(\frac{\tilde{z}}{n(\tilde{z},\alpha_i)}\right)^{1+\alpha_i}}{1+\alpha_i}$, $n(\tilde{z},\alpha_i)=\left(\frac{\tilde{z}^{\alpha_i}}{1-T'(\tilde{z})}\right)^{\frac{1}{1+\alpha_i}}$ and

$$\frac{\partial n(z^*,\alpha_i)}{\partial z^*}\Big|_{z^*=\tilde{z}}=\frac{\alpha_i}{1+\alpha_i}\left(\tilde{z}(1-T'(\tilde{z}))\right)^{\frac{-1}{1+\alpha_i}}+\frac{1}{1+\alpha_i}\tilde{z}^{\frac{\alpha_i}{1+\alpha_i}}\left(1-T'(\tilde{z})\right)^{\frac{-2-\alpha_i}{1+\alpha_i}}T''(\tilde{z})$$

Finally, we need to specify the twice continuously differentiable densities for each type, $f(n|\alpha_1),f(n|\alpha_2)$, along with the proportions of each type in society, $p(\alpha_1),p(\alpha_2)$. Given that one of the goals of this numerical exercise is to highlight that our simulation method can handle the possibility of individuals with multiple optima, we choose reasonable productivity densities that will likely generate jumping behavior. To do so, we note that in order to generate an individual with multiple optima, we need marginal tax rates to decrease over a certain portion of the income distribution (so that the consumption schedule is, in certain portions, convex) and we need indifference curves for one of the types to not be overly steep (as in Figure 3). Thus, for the type with steeper indifference curves (type α_2), we select $f(n|\alpha_2)$ such that there is a region of n's in which the density is increasing quickly. Assuming $p(\alpha_2)$ is large enough, this will generate tax rates that decline quickly over the given region. Consequently, this will induce some individual with type α_1 (who has less steep indifference curves) to have multiple optimal incomes. The densities we choose for each type are depicted in Figure 5 below. Note, the distribution $f(n|\alpha_1)$ is log-normal, whereas the distribution $f(n|\alpha_2)$ is constructed as a cubic spline (thus twice continuously differentiable) which is single peaked and has a region in which the density is increasing sharply. We choose the distributions so that the minimum and maximum incomes chosen in society are the same for type α_1 and α_2 populations under the optimal tax schedule. Finally, we will vary the proportions of types in society to highlight that at high values of $p(\alpha_2)$, type α_1 will have multiple optima. See Appendix C.1 for a step-by-step procedure of the simulation methodology for this example.

[38]By setting $W(u;n,\alpha)=\log(u)$, we assume that the government seeks to maximize total (log) utility without regard to whether income inequality is driven by productivity differences or differences in other dimensions. This welfare function therefore does not incorporate the notion of preference neutrality as discussed in, for example, Lockwood and Weinzierl (2016).

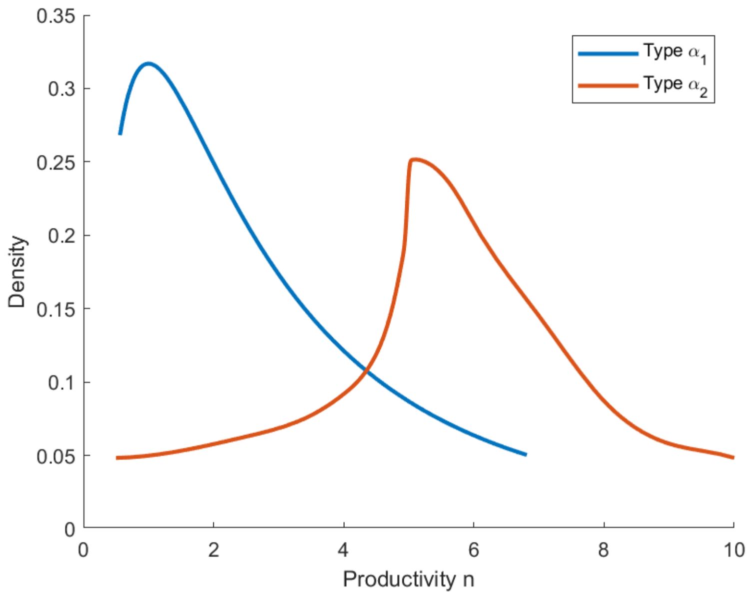

Figure 5: Conditional Productivity Densities, $f(n|\alpha_1)$ and $f(n|\alpha_2)$

5.3 Simulation Results

We now present simulation results for our numerical example. The first goal of our simulation exercise is to show that in settings with unidimensional heterogeneity, our simulation method yields identical tax schedules to the standard Mirrlees Hamiltonian method. As can be seen in Figure 6, optimal tax rates computed using these two methods are identical for the case in which all individuals are type 1 and the case in which all individuals are type 2. Next, note that when all individuals are type 2, marginal tax rates decrease sharply over incomes in the approximate range $(4, 7)$. This sharp decline is driven by the sharp increase in the productivity density $f(n|\alpha_2)$ over the approximate productivity range $(4, 5)$. Marginal tax rates then flatten out right after the peak of $f(n|\alpha_2)$ as the density begins to decline at a much slower rate than which it increased (refer back to Figure 5 for a depiction of $f(n|\alpha_2)$).[39] Finally, given this is a numerical example to illustrate our simulation method as opposed to a calibration exercise, the scale of incomes in Figure 6 (and all of our other simulation figures) does not hold meaning.

Next, we use our method to compute the optimal tax schedule when the population consists of both type 1 and type 2 agents. In Figure 7, we show how the tax schedule changes as we vary the percentage of type 2 individuals from 0% to 100%.[40] Unsurprisingly, for most of the income distribution, the optimal tax rate for x% of type 2 individuals lies somewhere in between the optimal tax rate when 0% of the population is type 2 and when 100% of the population is type 2. Most importantly, we find that for the three dotted schedules (50% type 2, 75% type 2, and

[39] While it is not immediately obvious from Figure 6, both of the marginal tax schedules shown are everywhere continuously differentiable.

[40] Figure 10 in Appendix C.2 plots the optimal consumption schedules for various percentages of type 2 individuals.

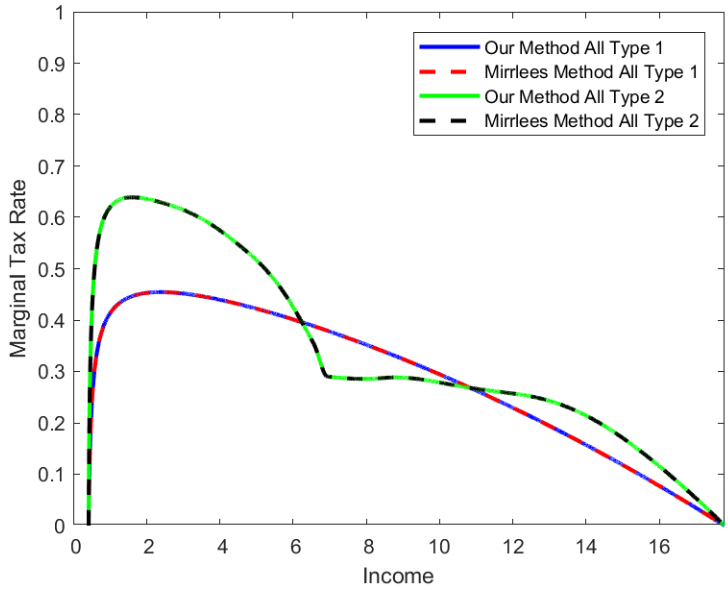

Figure 6: Optimal Tax Schedules Computed Using Equation 13 and Mirrlees's Method

90% type 2), there is a type 1 individual with multiple optimal income levels.[41] This can be seen in Figure 8 where we plot the productivity level n that chooses each income level under the optimal tax schedule, separately for α_1 types and α_2 types, when 50% of the population is type 2. In Figure 8, type α_1 individuals do not choose incomes between approximately 6.2 and 6.7, which indicates that, under the optimal tax schedule, there exists a type α_1 individual with multiple optimal incomes at approximately 6.2 and 6.7.

Another important feature to note from Figure 8 is that it is possible to have an individual with multiple optimal incomes yet no "missing mass" in the income distribution (as while no α_1 type chooses an income in the approximate range $(6.2, 6.7)$, the α_2 type does choose incomes in this range). This is important for reconciling the existence of individuals with multiple optima with the lack of a missing density anywhere in the observed income distribution.

Ultimately, there are three key takeaways from this simulation exercise: (1) our simulation method matches the results from the Mirrlees's Hamiltonian method if individuals only differ on the productivity dimension, (2) our method allows us to simulate optimal income tax schedules when agent heterogeneity is multidimensional and is able to handle the possibility that some individuals have multiple optimal income levels, and (3) optimal tax schedules can, in practice, lead to individuals having multiple optimal incomes when agent heterogeneity is multidimensional if marginal tax rates decrease relatively quickly over some portion of the income distribution.

[41]Commensurate with Proposition 5, the marginal tax schedule is discontinuous at each income level where an individual has multiple optimal incomes. However, the discontinuities in marginal tax rates are small ($\approx 1\%$) so are not readily apparent in Figure 7.

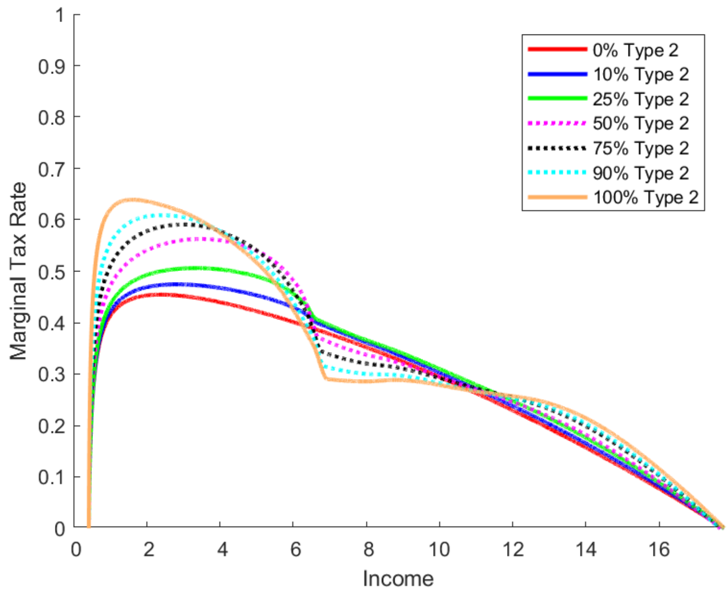

Figure 7: Optimal Tax Schedules for Various Percentages of Type 2 Individuals

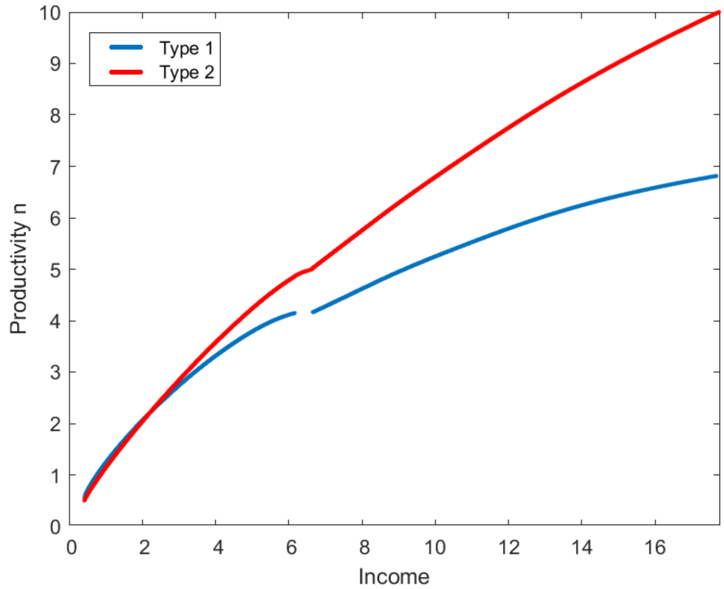

Figure 8: Productivity vs. Income when 50% of the Population is Type α_2

6 Conclusion

This paper has developed a theory of optimal income taxation when agents have multiple dimensions of heterogeneity using a perturbation method that mathematically formalizes the approach from Saez (2001). Once we move to a setting with multidimensional heterogeneity, we must consider the possibility that some individuals have multiple optimal incomes under the optimal tax schedule; this leads to individuals jumping around the tax schedule in response to small perturbations of the tax schedule. By explicitly accounting for these "jumping effects", we characterize the optimal tax schedule when agent heterogeneity is multidimensional. We then provide a partial characterization of when individuals will have multiple optimal incomes under the optimal tax schedule. In particular, we prove that this phenomenon cannot occur with unidimensional agent heterogeneity and for a specific form of multidimensional heterogeneity. However, we also prove that individuals with multiple optimal incomes can exist under sensible conditions with multidimensional agent heterogeneity; hence it is, in general, impossible to rule out individuals with multiple optimal incomes with multidimensional heterogeneity.

Finally, we develop a new methodology to simulate optimal income tax schedules that can be applied with multidimensional agent heterogeneity. This method is characterized by a differential equation governing the evolution of the tax schedule over the income distribution along with a search over the individuals that have multiple optimal income levels as well as the size of the discontinuities in marginal tax rates at their optimal income levels. We implement this method for a particular numerical example, showing that our method can solve for tax schedules when agent heterogeneity is multidimensional and can handle the possibility that some individuals have multiple optimal income levels.

Moving forward, this paper suggests a number of avenues for future research. First, the techniques employed in this paper can be used to solve other screening problems with multidimensional type spaces (but with unidimensional action and policy spaces), such as non-linear pricing. Further, we speculate that analyzing jumping effects may help towards solving even more complex problems with both multidimensional type spaces and multidimensional action and policy spaces. Second, it is important to better understand how jumping effects manifest in practice. While there is preliminary evidence that jumping behavior is empirically relevant, e.g., Rios (2019), further evidence is needed on the practical importance of jumping effects. Finally, towards utilizing the framework developed in this paper to better understand tax policy, further research is needed on the extent to which various forms of heterogeneity impact income differences as well as a better understanding of society's normative stance on the desirability of redistribution when inequality is driven by different forms of heterogeneity.

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A Proofs Appendix

A.1 Proof of Lemma 1

Proof. First, we show that (SCP) $\implies z(n,\alpha)$ is non-decreasing in n. Towards a contradiction, suppose not: $\exists n_3 < n_2$ s.t. $z_3 > z_2$, where $z_3 = z^*(n_3), z_2 = z^*(n_2)$ (note, α is held fixed so, for ease of notation, we will omit α as an argument). We thus have: $u\left(c(z_2), \frac{z_2}{n_2}\right) \geq u\left(c(z_3), \frac{z_3}{n_2}\right)$ and $u\left(c(z_2), \frac{z_2}{n_3}\right) \leq u\left(c(z_3), \frac{z_3}{n_3}\right)$. By continuity of u in n (and hence continuity of $u\left(c(z_2), \frac{z_2}{n}\right) - u\left(c(z_3), \frac{z_3}{n}\right)$ in n), we have that $\exists\ n_3 \leq \overline{n} \leq n_2$ s.t. $u\left(c(z_2), \frac{z_2}{\overline{n}}\right) = u\left(c(z_3), \frac{z_3}{\overline{n}}\right)$. Next, consider the function $\hat{c}(z;\bar{n},\bar{u})$ which denotes the consumption level s.t. type $\bar{n}$ has utility $\bar{u}$ at income level z, where $\bar{u} = u\left(c(z_2), \frac{z_2}{\overline{n}}\right) = u\left(c(z_3), \frac{z_3}{\overline{n}}\right)$. In other words, $\hat{c}(z;\bar{n},\bar{u})$ implicitly solves $u\left(\hat{c}, \frac{z}{\bar{n}}\right) = \bar{u}$. By construction, $\hat{c}(z_2) = c(z_2)$ and $\hat{c}(z_3) = c(z_3)$ (where, for ease of notation, we omit $\bar{n}, \bar{u}$ as arguments from $\hat{c}$). Now consider the following derivative:

$$\frac{\frac{\partial}{\partial z} u\left(\hat{c}(z), \frac{z}{\overline{n}}\right)}{u_c\left(\hat{c}(z), \frac{z}{\overline{n}}\right)} = \hat{c}'(z) + \frac{u_l\left(\hat{c}(z), \frac{z}{\overline{n}}\right)}{\overline{n} u_c\left(\hat{c}(z), \frac{z}{\overline{n}}\right)} = 0$$

This derivative is equal to zero, since $u\left(\hat{c}(z), \frac{z}{\overline{n}}\right) = \bar{u}\ \forall\ z$. Next, WLOG suppose $n_2 > \overline{n}$ (the argument is easily adapted if instead $\overline{n} > n_3$). Then, by the SCP we know that:

$$0 = \hat{c}'(z) + \frac{u_l\left(\hat{c}(z), \frac{z}{\overline{n}}\right)}{\overline{n} u_c\left(\hat{c}(z), \frac{z}{\overline{n}}\right)} < \hat{c}'(z) + \frac{u_l\left(\hat{c}(z), \frac{z}{n_2}\right)}{n_2 u_c\left(\hat{c}(z), \frac{z}{n_2}\right)}$$

Multiplying both sides of the inequality by $u_c\left(\hat{c}(z), \frac{z(s)}{n_2}\right)$ and integrating over $[z_2, z_3]$, we get:

$$\begin{aligned} 0 &< \int_{z_2}^{z_3} \left[\hat{c}'(z) u_c\left(\hat{c}(z), \frac{z}{n_2}\right) + \frac{1}{n_2} u_l\left(\hat{c}(z), \frac{z}{n_2}\right)\right] dz \\ &= u\left(\hat{c}(z_3), \frac{z_3}{n_2}\right) - u\left(\hat{c}(z_2), \frac{z_2}{n_2}\right) \\ &= u\left(c(z_3), \frac{z_3}{n_2}\right) - u\left(c(z_2), \frac{z_2}{n_2}\right) \end{aligned}$$

a contradiction.

Second, we show that (SCP) $\implies z^*(n,\alpha)$ is increasing in n whenever $T'(z)$ exists. Whenever $T'(z)$ exists, optimal income $z^*(n,\alpha)$ must satisfy agents' FOCs:

$$u_c\left(z^* - T(z^*), \frac{z^*}{n}; \alpha\right)(1 - T'(z^*)) - \frac{1}{n} u_l\left(z^* - T(z^*), \frac{z^*}{n}; \alpha\right) = 0$$

Now suppose that for $n_1 < n_2$, $z^*(n_1;\alpha) = z^*(n_2;\alpha) = z^*$. By the FOCs for n_1 and n_2 we know:

$$\frac{\frac{1}{n_1} u_l\left(z^* - T(z^*), \frac{z^*}{n_1}; \alpha\right)}{u_c\left(z^* - T(z^*), \frac{z^*}{n_1}; \alpha\right)} = \frac{\frac{1}{n_2} u_l\left(z^* - T(z^*), \frac{z^*}{n_2}; \alpha\right)}{u_c\left(z^* - T(z^*), \frac{z^*}{n_1}; \alpha\right)}$$

However, this violates the (SCP), a contradiction. Given Lemma 1, it must be the case that $z^*(n, \alpha)$ is increasing in n whenever $T'(z^*)$ exists.

□

A.2 Proof of Lemma 2

Proof. Suppose (SCP) holds, and fix α (we will omit α as an argument for ease of notation). Further, suppose under a given tax schedule an individual with productivity level n_1 has multiple optimal income levels. Denote her minimum optimal income level z_1^- and her maximum optimal income level z_1^+, where $z_1^+ > z_1^-$. We know by the proof of Lemma 1, that no other individual with $n_2 \neq n_1$ can have an optimal income level between z_1^-, z_1^+ by monotonicity of $z^*(n, \alpha)$ in n.

Next, by slightly adjusting the proof of Lemma 1, we can show that there cannot be multiple individuals with the same multiple optimal income levels. Towards a contradiction, suppose that individual n_2 has the same multiple optimal income levels as individual n_1. Thus, we have: $u\left(c(z_1^-), \frac{z_1^-}{n_2}\right) = u\left(c(z_1^+), \frac{z_1^+}{n_2}\right)$ and $u\left(c(z_1^-), \frac{z_1^-}{n_1}\right) = u\left(c(z_1^+), \frac{z_1^+}{n_1}\right)$. Next, consider $\hat{c}(z; n_1, \bar{u})$ which implicitly solves: $u\left(\hat{c}, \frac{z}{n_1}\right) = \bar{u}$ where $\bar{u} = u\left(c(z_1^-), \frac{z_1^-}{n_1}\right) = u\left(c(z_1^+), \frac{z_1^+}{n_1}\right)$, i.e., $\hat{c}(z; n_1, \bar{u})$ gives us the consumption level at each income z s.t. type n_1 receives utility $\bar{u}$. By construction, $\hat{c}(z_1^-) = c(z_1^-)$ and $\hat{c}(z_1^+) = c(z_1^+)$ (where, for ease of notation, we have omitted the arguments n_1 and $\bar{u}$ from $\hat{c}(\cdot)$). WLOG, assume $n_2 > n_1$. Then, by the definition of $\hat{c}(z)$ and the SCP we know:

$$0 = \frac{\frac{\partial}{\partial z} u\left(\hat{c}(z), \frac{z}{n_1}\right)}{u_c\left(\hat{c}(z), \frac{z}{n_1}\right)} = \hat{c}'(z) + \frac{u_l\left(\hat{c}(z), \frac{z}{n_1}\right)}{n_1 u_c\left(\hat{c}(z), \frac{z}{n_1}\right)} < \hat{c}'(z) + \frac{u_l\left(\hat{c}(z), \frac{z}{n_2}\right)}{n_2 u_c\left(\hat{c}(z), \frac{z}{n_2}\right)}$$

Multiplying both sides of the inequality by $u_c\left(\hat{c}(z), \frac{z}{n_2}\right)$ and integrating over z from $[z_1^-, z_1^+]$, we get:

$$\begin{aligned}
0 &< \int_{z_1^-}^{z_1^+} \left[\hat{c}'(z) u_c\left(\hat{c}(z), \frac{z}{n_2}\right) + \frac{1}{n_2} u_l\left(\hat{c}(z), \frac{z}{n_2}\right)\right] dz \\
&= u\left(\hat{c}(z_1^+), \frac{z_1^+}{n_2}\right) - u\left(\hat{c}(z_1^-), \frac{z_1^-}{n_2}\right) \\
&= u\left(c(z_1^+), \frac{z_1^+}{n_2}\right) - u\left(c(z_1^-), \frac{z_1^-}{n_2}\right)
\end{aligned}$$

a contradiction.

Thus, we know the following: for a fixed α, if an individual has multiple optimal income levels, no other individual can locate in between her minimum and maximum optimal income levels, nor can another individual have the same set of optimal income levels. This means that for a fixed level of α: (a) if an individual does have multiple optimal incomes, there must be a jump discontinuity in the optimal income function $z^*(n)$ (the converse is also true: a jump discontinuity in $z^*(n)$ implies an individual must have multiple optimal incomes by continuity of the utility function) and (b) each of these jump discontinuities in $z^*(n)$ corresponds to a single type n who has multiple optima. Because $z^*(n)$ is weakly monotonic for each fixed α, we know that it can only have a countable number of such jump discontinuities. Thus, we can associate each jump discontinuity in $z^*(n)$ to a single type n who has multiple optimal incomes, which implies there are at most a countable number of individuals with multiple optima for each α.

Next, for each α, let denote $\{m_i(\alpha)\}$ denote the countable set of types m (indexed by i) which have multiple optimal income levels. Finally, the set of all individuals over $N \times A$ space that have multiple optima is given by $\cup_{\alpha \in A}\{m_i(\alpha)\}$, which is countable given the union of countable sets is countable.

□

A.3 Proof of Lemma 3

Proof. Suppose $T'(z)$ exists, meaning optimal income, $z^*(n, \alpha)$, satisfies the following condition:

$$FOC(z^*, n; \alpha) = 0$$

where

$$FOC(z, n; \alpha) = u_c\left(z - T(z), \frac{z}{n}; \alpha\right)(1 - T'(z)) + \frac{1}{n}u_l\left(z - T(z), \frac{z}{n}; \alpha\right)$$

Now, suppose $z^*(n, \alpha)$ is continuous at n_1, but $SOC(z^*(n_1; \alpha), n_1; \alpha) = 0$ (where $SOC(z, n; \alpha) = \frac{\partial FOC(z,n;\alpha)}{\partial z}$).[42] By continuity in z^* at n_1, we know for a small ϵ change in n, we have a corresponding small δ change in optimal income. However, the FOC at $(z + \delta, n_1 + \epsilon, \alpha)$ will be approximately equal to:

$$\begin{aligned} FOC(z + \delta, n_1 + \epsilon; \alpha) &= \left(FOC(z, n; \alpha) + \epsilon\frac{\partial FOC(z, n; \alpha)}{\partial n} + \delta\frac{\partial FOC(z, n; \alpha)}{\partial z}\right)\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} \\ &= \epsilon\frac{\partial FOC(z, n; \alpha)}{\partial n}\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} > 0 \end{aligned}$$

where the second equality comes from the fact that the SOC is 0 for (n_1, α) and we know that the FOC must be 0 for type (n_1, α) when evaluated at her optimal income, $z^*(n_1; \alpha)$. The last inequality comes from the fact that by (SCP), $\frac{\partial FOC}{\partial n}\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} > 0$ (see A.3.1 below).

Since $FOC(z + \delta, n_1 + \epsilon; \alpha) > 0$, $z + \delta$ cannot be the optimal income for type $(n_1 + \epsilon, \alpha)$ (as the optimal income must set the FOC to 0 when T' exists). Hence, the optimal income function $z^*(n, \alpha)$ cannot be continuous at n_1. In other words, if the SOC holds weakly for a given (n_1, α), there must be a jump discontinuity in the optimal income function $z^*(n, \alpha)$ at n_1, meaning that type (n_1, α) has multiple optimal income levels under the given tax function.

Finally, because we know that if an agent's $SOC = 0$ at an agent's optimal income level, then she has multiple optimal income levels, we also know that if an agent is to only have one

[42]Note, $\frac{\partial FOC(z,n;\alpha)}{\partial z}$ exists as we assume $T''(z)$ exists.

optimal income level, her $SOC < 0$ at this unique optimal income level.[43]

A.3.1 Showing $\frac{\partial FOC}{\partial n}\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} > 0$

Differentiating the FOC w.r.t. n gives:

$$\frac{\partial FOC}{\partial n} = -\frac{1}{n^2}\left(u_{cl}z(1-T') + u_{ll}\frac{z}{n} + u_l\right)$$

We will show by (SCP), that $\left(u_{cl}z(1-T') + u_{ll}\frac{z}{n} + u_l\right) < 0$ at $(z^*(n_1,\alpha),n_1,\alpha)$. The SCP gives us:

$$-\frac{1}{n^2u_c^2}\left(-u_cu_l - \frac{z}{n}u_{ll}u_c + \frac{z}{n}u_lu_{cl}\right) < 0$$

which gives

$$-u_cu_l - \frac{z}{n}u_{ll}u_c + \frac{z}{n}u_lu_{cl} > 0$$

At $(z^*(n_1,\alpha),n_1,\alpha)$, we know the FOC is 0, i.e., $-u_c(1-T') = \frac{1}{n}u_l$. Thus, we get:

$$\left(-u_cu_l - \frac{z}{n}u_{ll}u_c - zu_c(1-T')u_{cl}\right)\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} > 0$$

Dividing through by $-u_c$ we get:

$$\left(u_l + \frac{z}{n}u_{ll} + z(1-T')u_{cl}\right)\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} < 0$$

Thus we get $\frac{\partial FOC}{\partial n}\Big|_{(z^*(n_1,\alpha),n_1,\alpha)} > 0$. □

A.4 Full $\tau(z)$ Function

We define $\tau(z)$ to be a twice continuously differentiable. Hence, we need a $\tau(z)$ function that satisfies $(\tau(\tilde{z}-d\tilde{z}^2), \tau'(\tilde{z}-d\tilde{z}^2), \tau''(\tilde{z}-d\tilde{z}^2)) = (0,0,0)$, $(\tau(\tilde{z}), \tau'(\tilde{z}), \tau''(\tilde{z}) = (d\tilde{z}^2, 1, 0)$, $(\tau(\tilde{z}+d\tilde{z}), \tau'(\tilde{z}+d\tilde{z}), \tau''(\tilde{z}+d\tilde{z})) = (d\tilde{z}+d\tilde{z}^2, 1, 0)$, $(\tau(\tilde{z}+d\tilde{z}+d\tilde{z}^2), \tau'(\tilde{z}+d\tilde{z}+d\tilde{z}^2), \tau''(\tilde{z}+d\tilde{z}+d\tilde{z}^2) = (d\tilde{z}, 0, 0)$. One can check that the following $\tau(z)$ satisfies these conditions:

$$\begin{cases} \tau(z) = 0 & \text{if } z \leq \tilde{z} - d\tilde{z}^2 \\ \tau(z) = \tau_1(z) & \text{if } z \in [\tilde{z} - d\tilde{z}^2, \tilde{z}] \\ \tau(z) = z - \tilde{z} + d\tilde{z}^2 & \text{if } z \in [\tilde{z}, \tilde{z} + d\tilde{z}] \\ \tau(z) = \tau_2(z) & \text{if } z \in [\tilde{z} + d\tilde{z}, \tilde{z} + d\tilde{z} + d\tilde{z}^2] \\ \tau(z) = d\tilde{z} & \text{if } z \geq \tilde{z} + d\tilde{z} + d\tilde{z}^2 \end{cases}$$

[43]Note, if an income level satisfies $FOC(z,n;\alpha) = 0$ and $SOC(z,n;\alpha) < 0$, this does *not* imply that the z is an unique global maximum for type (n,α). Rather z could simply be a local maximum for type (n,α).

where we define:

$$\tau_1(z) = 3d\tilde{z}^2\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^5 - 8d\tilde{z}^2\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^4 + 6d\tilde{z}^2\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^3$$

$$\tau_2(z) = 9d\tilde{z}^2\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^5 - 22d\tilde{z}^2\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^4 + 14d\tilde{z}^2\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^3 + d\tilde{z}$$

A.4.1 Proof that $\tau(z)$ and $\tau'(z)$ Are Bounded on $[\tilde{z}-d\tilde{z}^2, \tilde{z}]$ and $[\tilde{z}-d\tilde{z}^2, \tilde{z}]$

It's immediately clear, that $\tau_1(z)$ and $\tau_2(z)$ are bounded by $d\tilde{z}^2$. As far as $\tau_1'(z)$ and $\tau_2'(z)$, note that:

$$\tau_1'(z) = 15\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^4 - 32\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^3 + 18\left(\frac{z-(\tilde{z}-d\tilde{z}^2)}{d\tilde{z}^2}\right)^2$$

$$\tau_2'(z) = -45\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^4 + 88\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^3 - 42\left(\frac{(\tilde{z}+d\tilde{z}+d\tilde{z}^2)-z}{d\tilde{z}^2}\right)^2$$

It's easily shown that both of these functions are bounded on their stated domains.

A.5 Proof of Proposition 1

We start by plugging our chosen $\tau(z)$ into Equation 5 (denoting the set $Z^+ \equiv (\tilde{z}+d\tilde{z}+d\tilde{z}^2, \infty) \setminus \{K_i\}$):[44]

$$\begin{aligned}
&\sum_{\alpha\in A}\int_{\tilde{z}-d\tilde{z}^2}^{\tilde{z}}\left[-W_u(u^*)u_c^*\tau(z^*) + \lambda\left(T'(z^*)\frac{\partial z^*}{\partial\mu}\Big|_{\mu=0} + \tau(z^*)\right)\right]dH(z^*|\alpha)p(\alpha)+\\
&\sum_{\alpha\in A}\int_{\tilde{z}}^{\tilde{z}+d\tilde{z}}\left[-W_u(u^*)u_c^*(z^*-\tilde{z}+d\tilde{z}^2) - \lambda\left(T'(z^*)\left(\frac{Z^c_{z^*,\alpha}z^*}{1-T'(z^*)} + \frac{\eta_{z^*,\alpha}}{1-T'(z^*)}(z^*-\tilde{z}+d\tilde{z}^2)\right) - (z^*-\tilde{z}+d\tilde{z}^2)\right)\right]dH(z^*|\alpha)p(\alpha)+\\
&\sum_{\alpha\in A}\int_{\tilde{z}+d\tilde{z}}^{\tilde{z}+d\tilde{z}+d\tilde{z}^2}\left[-W_u(u^*)u_c^*\tau(z^*) + \lambda\left(T'(z^*)\frac{\partial z^*}{\partial\mu}\Big|_{\mu=0} + \tau(z^*)\right)\right]dH(z^*|\alpha)p(\alpha)+\\
&\sum_{\alpha\in A}\int_{Z^+}\left[-W_u(u^*)u_c^*d\tilde{z} - \lambda\left(T'(z^*)\frac{\eta_{z^*,\alpha}}{1-T'(z^*)}d\tilde{z} - d\tilde{z}\right)\right]dH(z^*|\alpha)p(\alpha)+\\
&\sum_{K_i} d\tilde{z}\lambda\left(1-\overline{\omega}(K_i)\right)p_K(K_i)\mathbb{1}(K_i > \tilde{z}+d\tilde{z}+d\tilde{z}^2) = 0
\end{aligned} \tag{15}$$

We know that Equation 15 must hold for all values of $d\tilde{z}$ (as the derivative of the government Lagrangian in the direction of *any* function $\tau(z)$ must be 0). We first divide through by $d\tilde{z}$ everywhere and then take the limit of Equation 15 as $d\tilde{z} \to 0$. Doing so, we get the following:

[44] Note, individuals who bunch at kinks can only locate outside the interval $[\tilde{z}-d\tilde{z}^2, \tilde{z}+d\tilde{z}+d\tilde{z}^2]$ given our choice of $\tau(z)$.

$$-\sum_{\alpha\in A}\lambda\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}Z^c_{\tilde{z},\alpha}h(\tilde{z}|\alpha)p(\alpha)-\sum_{\alpha\in A}\int_{(\tilde{z},\infty)\setminus\{K_i\}}\lambda\eta_{z^*,\alpha}\frac{T'(z^*)}{1-T'(z^*)}dH(z^*|\alpha)p(\alpha)+$$
$$\sum_{\alpha\in A}\int_{(\tilde{z},\infty)\setminus\{K_i\}}(\lambda-W_u(u^*)u^*_c)\,dH(z^*|\alpha)p(\alpha)+\sum_{K_i}\lambda\left(1-\overline{\omega}(K_i)\right)p_K(K_i)\mathbb{1}(K_i>\tilde{z})=0 \tag{16}$$

The first term and third term of Equation 15 go to 0 as the integrals are of order $d\tilde{z}^2$ (the integrands are bounded and integrated over an interval of size $d\tilde{z}^2$)[45]; thus dividing these integrals by $d\tilde{z}$ leaves us with a term that is of order $d\tilde{z}$. Hence, these two terms go to zero as $d\tilde{z}$ goes to 0. It's easy to see how the fourth term of Equation 15 goes to the second and third terms of Equation 16 after dividing by $d\tilde{z}$ and taking the limit as $d\tilde{z}$ goes to 0. Similarly, it's easy to see how the fifth term of Equation 15 goes to the fourth term of Equation 16. For the second term of Equation 15 (after dividing through by $d\tilde{z}$), we use the rectangle approach to approximate the integral as:[46]

$$\sum_{\alpha\in A}\left[-W_u(u^*)u^*_c d\tilde{z}^2-\lambda\left(T'(\tilde{z})\left(\frac{Z^c\tilde{z}}{1-T'(\tilde{z})}+\frac{\eta}{1-T'(\tilde{z})}d\tilde{z}^2\right)-d\tilde{z}^2\right)\right]dH(\tilde{z}|\alpha)p(\alpha)\rightarrow$$
$$-\sum_{\alpha\in A}\lambda\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}Z^c dH(\tilde{z}|\alpha)p(\alpha)=-\sum_{\alpha\in A}\lambda\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}Z^c h(\tilde{z}|\alpha)p(\alpha)$$

Finally, denoting $\bar{Z}^c_{\tilde{z}}=\sum_{\alpha\in A}Z^c_{\tilde{z},\alpha}p(\alpha|\tilde{z})$ as the average compensated elasticity at income $\tilde{z}$, $\bar{\eta}_{z^*}=\sum_{\alpha\in A}\eta_{z^*,\alpha}p(\alpha|z^*)$ as the average income effect parameter at z^*, $\bar{\omega}(z^*)=\sum_{\alpha\in A}\frac{W_u(u^*)u^*_c}{\lambda}p(\alpha|z^*)$ as the average social welfare weight at income z^*, switching the order of the sum and the integral on the second term in Equation 16, and noting that

[45] As the utility function is clearly bounded in this range, it suffices to show that $\tau(z)$ and $\tau'(z)$ are bounded, which implies that $\frac{\partial z^*}{\partial\mu}|_{\mu=0}$ is bounded as well. See Appendix A.4.1.

[46] In order to apply the rectangle approach, it suffices to ensure that the integrand is continuous and that $H(\tilde{z}|\alpha)$ is continuously differentiable at $\tilde{z}$. Given that Equation 15 only holds at non-kink points, we only consider $\tilde{z}$ where $T(\tilde{z})$ is twice continuously differentiable. Thus, the integrand is continuous by our assumptions on utility. If $n(\tilde{z},\alpha)$ is on the boundary of support of $F(n|\alpha)$, $F'(n|\alpha)\neq 0$, and $\tilde{z}$ is an interior income chosen in society, then there will be a kink point in the tax schedule at $\tilde{z}$ meaning Equation 15 does not apply (note, we are ignoring knife-edge cases wherein there's a $\tilde{z}$ for which the support of $F(n|\alpha_i)$ ends and the support of $F(n|\alpha_j)$ begins and these effects perfectly offset so that the tax schedule is differentiable). Hence, we can restrict attention to cases where $n(\tilde{z},\alpha)$ is in the interior of the support of $F(n|\alpha)$. In this case, $H(\tilde{z}|\alpha)$ is continuously differentiable so long as $F(n|\alpha)$ is continuously differentiable (which we assume) and $\frac{\partial n(z;\alpha)}{\partial z}=\left(\frac{\partial z(n,\alpha)}{\partial n}\right)^{-1}$ exists and is continuous. $\frac{\partial z(n,\alpha)}{\partial n}$ exists as long as the SOC holds strictly, which is true by Lemma 3 as we assume no one has multiple optima at $\tilde{z}$. Finally, $\frac{\partial z(n,\alpha)}{\partial n}>0$ by Lemma 1 so that $\frac{\partial n(z;\alpha)}{\partial z}$ exists; continuity of $\frac{\partial n(z;\alpha)}{\partial z}$ follows from our assumptions of twice continuous differentiability on the utility functions and tax schedule at non-kink points. Hence, $H(\tilde{z}|\alpha)$ will be continuously differentiable at all non-kink points of the tax schedule, so that it is admissible to use the rectangle approach on Equation 15.

$h(z|\alpha)p(\alpha) = p(\alpha|z)h(z)$, we can rewrite Equation 16 as follows:

$$\int_{\tilde{z}}^{\infty} (1 - \bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}\tilde{z}\bar{Z}_{\tilde{z}}^{c} h(\tilde{z}) - \int_{(\tilde{z},\infty)\setminus\{K_i\}} \bar{\eta}_{z^*} \frac{T'(z^*)}{1 - T'(z^*)} dH(z^*) = 0$$

A.6 Proof Equation 8 Holds At Points of Non-Differentiability

Proof. We seek to show Equation 8 (reproduced below) holds even at points of non-differentiability of the tax schedule.

$$\left.\frac{\partial m(\alpha)}{\partial \mu}\right|_{\mu=0} = -\frac{u_c\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)\tau(z^{*-}) - u_c\left(c(z^{*+}), \frac{z^{*+}}{m}; \alpha\right)\tau(z^{*+})}{u_l\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)\frac{z^{*-}}{m^2} - u_l\left(c(z^{*+}), \frac{z^{*+}}{m}; \alpha\right)\frac{z^{*+}}{m^2}}$$

Recall that $m(\alpha)$ is the individual with multiple optima and that $m(\alpha)$, $z^{*-}(\alpha)$, and $z^{*+}(\alpha)$ are functions of the tax schedule. Hence, we now explicitly write these as functions of μ: $m(\alpha,\mu)$, $z^{*-}(\alpha,\mu)$, and $z^{*+}(\alpha,\mu)$, which satisfying the following indifference relation:

$$\begin{aligned} &u\left(z^{*-}(\alpha,\mu) - T(z^{*-}(\alpha,\mu)) - \mu\tau(z^{*-}(\alpha,\mu)), \frac{z^{*-}(\alpha,\mu)}{m(\alpha,\mu)}; \alpha\right) \\ &= u\left(z^{*+}(\alpha,\mu) - T(z^{*+}(\alpha,\mu)) - \mu\tau(z^{*+}(\alpha,\mu)), \frac{z^{*+}(\alpha,\mu)}{m(\alpha,\mu)}; \alpha\right) \end{aligned} \tag{17}$$

WLOG, let us assume that the tax schedule is non-differentiable only at z^{*-} and not at z^{*+}. Denote the right marginal tax rate at z^{*-} as T'^r and the left marginal tax rate at z^{*-} as T'^l. There are a number of cases to consider. First, suppose that for the indifferent individual $m(\alpha, 0)$:

$$u_c\left(c(z^{*-}(\alpha,0)), \frac{z^{*-}(\alpha,0)}{m(\alpha,0)}; \alpha\right)(1 - T'^r) + u_l\left(c(z^{*-}(\alpha,0)), \frac{z^{*-}(\alpha,0)}{m(\alpha,0)}; \alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2} < 0 \tag{18}$$

and

$$u_c\left(c(z^{*-}(\alpha,0)), \frac{z^{*-}(\alpha,0)}{m(\alpha,0)}; \alpha\right)(1 - T'^l) + u_l\left(c(z^{*-}(\alpha,0)), \frac{z^{*-}(\alpha,0)}{m(\alpha,0)}; \alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2} > 0 \tag{19}$$

Note, in this case, for sufficiently small μ, Equations 18 and 19 both still hold, so that z^{*-} does not change with μ. In this case we can consider z^{*-} a constant (as a function of μ) and apply the implicit function theorem to Equation 17 to yield:

$$u_c\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\tau(z^{*-})-u_c\left(c(z^{*+}),\frac{z^{*+}}{m};\alpha\right)\tau(z^{*+})+$$
$$\left[u_l\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2}-u_l\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*+}(\alpha,0)}{m(\alpha,0)^2}\right]\frac{\partial m(\alpha,\mu)}{\partial\mu}\Big|_{\mu=0}+$$
$$\left[u_c\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)(1-T'(z^{*+}))+u_l\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*+}(\alpha,0)}{m(\alpha,0)^2}\right]\frac{\partial z^{*+}(\alpha,\mu)}{\partial\mu}\Big|_{\mu=0}=0 \tag{20}$$

Because we have:

$$u_c\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)(1-T'(z^{*+}))+u_l\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*+}(\alpha,0)}{m(\alpha,0)^2}=0$$

we can rearrange Equation 20 to yield Equation 8. Now, suppose instead that:

$$u_c\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)(1-T'^r)+u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)\frac{z^{*-}}{m^2}=0 \tag{21}$$

and

$$u_c\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)(1-T'^l)+u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)\frac{z^{*-}}{m^2}>0 \tag{22}$$

Now, consider perturbing the tax schedule in the direction of a function $\tau(z)$ that induces individual $m(\alpha,0)$ to remain at income level $z^{*-}(\alpha,0)$. In this case, we can apply the same logic as before, holding z^{*-} constant as a function of μ to yield Equation 8 as our expression for $\frac{\partial m(\alpha)}{\partial\mu}\Big|_{\mu=0}$. On the other hand, suppose that perturbing the tax schedule in the direction of a function $\tau(z)$ induces $m(\alpha,0)$ to move along the tax schedule to the right. In this case, when we apply the implicit function theorem to Equation 17 we get:

$$u_c\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\tau(z^{*-})-u_c\left(c(z^{*+}),\frac{z^{*+}}{m};\alpha\right)\tau(z^{*+})+$$
$$\left[u_l\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2}-u_l\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*+}(\alpha,0)}{m(\alpha,0)^2}\right]\frac{\partial m(\alpha,\mu)}{\partial\mu}\Big|_{\mu=0}+$$
$$\left[u_c\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)(1-T'^r)+u_l\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2}\right]\frac{\partial z^{*-}(\alpha,\mu)}{\partial\mu}\Big|_{\mu=0}+$$
$$\left[u_c\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)(1-T'(z^{*+}))+u_l\left(c(z^{*+}(\alpha,0)),\frac{z^{*+}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*+}(\alpha,0)}{m(\alpha,0)^2}\right]\frac{\partial z^{*+}(\alpha,\mu)}{\partial\mu}\Big|_{\mu=0}=0 \tag{23}$$

But now, we also have that:

$$u_c\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)(1-T'^r)+u_l\left(c(z^{*-}(\alpha,0)),\frac{z^{*-}(\alpha,0)}{m(\alpha,0)};\alpha\right)\frac{z^{*-}(\alpha,0)}{m(\alpha,0)^2}=0$$

Hence, as before, we can rearrange to yield Equation 8. The logic is identical if instead we have:

$$u_c\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)(1-T'^r) + u_l\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)\frac{z^{*-}}{m^2} < 0$$

and

$$u_c\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)(1-T'^l) + u_l\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right)\frac{z^{*-}}{m^2} = 0$$

Moreover, the logic is identical if the tax schedule is instead (or also) non-differentiable at z^{*+}. Hence, Equation 8 holds even if the tax schedule is non-differentiable at z^{*-} and/or z^{*+}.

□

A.7 Proof of Proposition 2

First, we can write the government's Lagrangian as:[47]

$$\begin{aligned}\mathcal{L} = &\sum_{\alpha\in A}\int_0^{m_1(\alpha)}\left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T\left(z^*(n,\alpha)\right)\right] dF(n|\alpha)p(\alpha)+\\ &\sum_{\alpha\in A}\sum_{i=1}^{M(\alpha)-1}\int_{m_i(\alpha)}^{m_{i+1}(\alpha)}\left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T\left(z^*(n,\alpha)\right)\right] dF(n|\alpha)p(\alpha)+\\ &\sum_{\alpha\in A}\int_{m_{M(\alpha)}(\alpha)}^{\infty}\left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T\left(z^*(n,\alpha)\right)\right] dF(n|\alpha)p(\alpha)\end{aligned}$$

Taking the derivative of the Lagrangian w.r.t. μ and evaluating at $\mu = 0$ we get an augmented version of Equation 9:

$$\begin{aligned}&\sum_{\alpha\in A}\int_0^{\infty}\left[-W_u(u^*)u_c^*\tau(z^*) + \lambda\left(\frac{\partial T(z^*)}{\partial\mu}\Big|_{\mu=0} + \tau(z^*)\right)\right] dF(n|\alpha)p(\alpha)+\\ &\sum_{\alpha\in A}\sum_{i=1}^{M(\alpha)}\lambda\left(T\left(z_i^{*-}(\alpha)\right) - T\left(z_i^{*+}(\alpha)\right)\right)\frac{\partial m_i(\alpha)}{\partial\mu}\Big|_{\mu=0} f(m_i(\alpha)|\alpha)p(\alpha) = 0\end{aligned} \tag{24}$$

Now consider the same perturbation as in Subsection 3.2. For the $(m_i(\alpha), \alpha)'s$ with $z_i^{*+}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$ and $z_i^{*-}(\alpha) < \tilde{z} - d\tilde{z}^2$, the jumping effect is equal to:

$$\lambda\left(T\left(z_i^{*-}(\alpha)\right) - T\left(z_i^{*+}(\alpha)\right)\right)\frac{u_{ci}^{*+}(\alpha)\, d\tilde{z}}{u_{li}^{*-}(\alpha)\frac{z_i^{*-}(\alpha)}{m_i(\alpha)^2} - u_{li}^{*+}(\alpha)\frac{z_i^{*+}(\alpha)}{m_i(\alpha)^2}} f(m_i(\alpha)|\alpha) \equiv J_{1i}(\alpha)\lambda d\tilde{z}$$

where $u_{ci}^{*+} = u_c\left(z_i^{*+}(\alpha) - T\left(z_i^{*+}(\alpha)\right), \frac{z_i^{*+}(\alpha)}{m_i(\alpha)}; \alpha\right)$ etc.

[47] We've used Assumption 2 to ensure that the set $\{m_i(\alpha)\}$ can be totally ordered using the usual relation $<$, so that we can write out the Lagrangian as a sum over integrals with endpoints in $\{m_i(\alpha)\}$.

On the other hand, for the $(m_i(\alpha), \alpha)'s$ with $z_i^{*-}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$, the jumping effect is equal to:

$$\lambda\left(T\left(z_i^{*-}(\alpha)\right) - T\left(z_i^{*+}(\alpha)\right)\right) \frac{u_{ci}^{*+}(\alpha)\, d\tilde{z} - u_{ci}^{*-}(\alpha)\, d\tilde{z}}{u_{li}^{*-}(\alpha)\frac{z_i^{*-}(\alpha)}{m_i(\alpha)^2} - u_{li}^{*+}(\alpha)\frac{z_i^{*+}(\alpha)}{m_i(\alpha)^2}} f(m_i(\alpha)|\alpha) \equiv J_{2i}(\alpha)\lambda d\tilde{z}$$

Dividing Equation 24 by $d\tilde{z}\lambda$ and taking the limit as $d\tilde{z} \to 0$ (following the same ideas as in Appendix A.5) we get:

$$\int_{\tilde{z}}^{\infty} (1 - \bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}\tilde{z}\bar{Z}^c_{\tilde{z}}h(\tilde{z}) - \int_{(\tilde{z},\infty)\backslash\{K_i\}} \bar{\eta}_{z^*}\frac{T'(z^*)}{1 - T'(z^*)}dH(z^*)$$
$$\sum_{\alpha\in A}\sum_{i=1}^{M(\alpha)}\left[J_{1i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha) < \tilde{z} < z_i^{*+}(\alpha)\right) + J_{2i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha) > \tilde{z}\right)\right] p(\alpha) = 0$$

A.8 Equivalence of Equation 11 to Mirrlees (1971) With Unidimensional Heterogeneity

We show equivalence of Equation 11 to Mirrlees's (1971) optimality condition when heterogeneity exists only in the productivity dimension. We prove this for the case where just one individual has multiple optima; the proof is easily extended to the case where more individuals have multiple optima.[48] Suppose some individual with productivity m has multiple optimal income level z^{*-} and z^{*+}. We will show equivalence of Equation 11 at all income levels $\tilde{z}$ for which the tax schedule is differentiable and some individual $n \neq m$ chooses to locate at $\tilde{z}$.[49] We start by considering incomes $\tilde{z} > z^{*+}$; we know Equation 11 simplifies to:[50]

$$\int_{\tilde{z}}^{\infty}\left[1 - \omega(z^*) - \frac{T'(z^*)}{1 - T'(z^*)}\eta_{z^*}\right] dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) = 0 \tag{25}$$

What does Equation 11 look like for incomes below z^{*-}? In this case, for incomes at

[48] Saez (2001) showed that Equation 11 is equivalent to Mirrlees's (1971) optimality condition if all individuals have a unique optima.

[49] Equation 11 only holds at a given $\tilde{z}$ when $T(\tilde{z})$ is differentiable and when no individual has multiple optima at $\tilde{z}$. Moreover, while Equation 11 can be applied at incomes between z^{*-} and z^{*+}, it will not generate an optimality condition for marginal tax rates between (z^{*-}, z^{*+}) as no one locates in this range; rather it will generate an optimality condition that $(T(z^{*+}) - T(z^{*-}))$ must satisfy (this will become clearer later on in the proof). Similarly, Mirrlees's (1971) optimality condition does not specify marginal tax rates between z^{*-} and z^{*+}. Ultimately, this is not overly important because tax rates between z^{*-} and z^{*+} are not unique: any tax schedule that generates a consumption schedule that lies below type m's indifference curve and leaves type m indifferent between z^{*-} and z^{*+} yields the same total welfare.

[50] We abuse notation and define $\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*} \equiv 0$ if the tax schedule is non-differentiable at z^*. This notational simplification means that we do not have to explicitly exclude kink points from the integrals as in Equation 11.

which the tax schedule is differentiable, Equation 11 yields:

$$\int_{\tilde{z}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]dH(z^*)-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})+J_2=0 \tag{26}$$

where

$$J_2=\left(T(z^{*-})-T(z^{*+})\right)\frac{u_c\left(c(z^{*+}),\frac{z^{*+}}{m}\right)-u_c\left(c(z^{*-}),\frac{z^{*-}}{m}\right)}{\frac{1}{m^2}\left(u_l\left(c(z^{*-}),\frac{z^{*-}}{m}\right)z^{*-}-u_l\left(c(z^{*+}),\frac{z^{*+}}{m}\right)z^{*+}\right)}f(m)$$

It will be helpful to rewrite Equation 26 by considering Equation 11 for incomes between z^{*-} and z^{*+}. By the SCP, no individuals will locate between z^{*-} and z^{*+} in the unidimensional world; thus, marginal tax rate are not uniquely pinned down between z^{*-} and z^{*+} as any tax schedule that generates a consumption schedule that lies below type m's indifference curve and keeps type m indifferent between z^{*-} and z^{*+} leads to the same total welfare. However, we can nonetheless apply Equation 11 to incomes between z^{*-} and z^{*+} to get a condition that pins down $(T(z^{*-})-T(z^{*+}))$:=

$$\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]dH(z^*)+J_1=0 \tag{27}$$

where

$$J_1=\left(T(z^{*-})-T(z^{*+})\right)\frac{u_c\left(c(z^{*+}),\frac{z^{*+}}{m}\right)}{\frac{1}{m^2}\left(u_l\left(c(z^{*-}),\frac{z^{*-}}{m}\right)z^{*-}-u_l\left(c(z^{*+}),\frac{z^{*+}}{m}\right)z^{*+}\right)}f(m)$$

Note, there are no elastic effects of changing tax rates between (z^{*-},z^{*+}) as no one locates in this range. Hence, Equation 27 does not pin down marginal tax rates between (z^{*-},z^{*+}); rather, Equation 27 provides a condition that pins down $(T(z^{*-})-T(z^{*+}))$. Next we note that:

$$\begin{aligned}J_2&=\frac{u_c\left(c(z^{*+}),\frac{z^{*+}}{m}\right)-u_c\left(c(z^{*-}),\frac{z^{*-}}{m}\right)}{u_c\left(c(z^{*+}),\frac{z^{*+}}{m}\right)}J_1\\&=-\left(1-\frac{u_c^{*-}}{u_c^{*+}}\right)\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]dH(z^*)\end{aligned}$$

where $u_c^{*+}=u_c(z^{*+}-T(z^{*+}),z^{*+}/m)$ and $u_c^{*-}=u_c(z^{*-}-T(z^{*-}),z^{*-}/m)$. We can use this relationship to rewrite Equation 26 for all incomes $\tilde{z}<z^{*-}$ for which the tax schedule

is differentiable:[51]

$$\int_{\tilde{z}}^{z^{*-}} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1 - T'(z^*)}\eta_{z^*}\right] dH(z^*) + \frac{u_c^{*-}}{u_c^{*+}} \int_{z^{*+}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1 - T'(z^*)}\eta_{z^*}\right] dH(z^*)$$
$$- \frac{T'(\tilde{z})}{1 - T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) = 0 \tag{28}$$

Thus, in order to show that Equation 11 is identical to Mirrlees's (1971) optimality condition at all income levels $\tilde{z}$ for which the tax schedule is differentiable and some individual $n \neq m$ chooses to locate at $\tilde{z}$, it suffices to show that Equation 25 and Equation 28 are equivalent to Mirrlees's (1971) optimality condition.

We can combine Equations 25 and 28 by defining the following function $Q(z)$: $Q(z) \equiv \frac{u_c^{*-}}{u_c^{*+}}$ if $z \geq z^{*+}$, $Q(z) \equiv 1$ if $z \leq z^{*-}$, and $Q(z) \equiv 0$ if $z^{*-} < z < z^{*+}$. The optimal tax schedule must satisfy the following differential equation $\forall\ \tilde{z} \notin \{z^{*-}, z^{*+}\}$ where the tax schedule is differentiable (recognizing that Equation 29 holds vacuously between z^{*-} and z^{*+} as $h(\tilde{z}) = 0$ for $\tilde{z} \in (z^{*-}, z^{*+})$):

$$\int_{\tilde{z}}^{\infty} Q(z^*) \left[1 - \omega(z^*) - \frac{T'(z^*)}{1 - T'(z^*)}\eta_{z^*}\right] dH(z^*) - Q(\tilde{z}) \frac{T'(\tilde{z})}{1 - T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) = 0 \tag{29}$$

Changing variables from z^* to n, denoting the individual who chooses $\tilde{z}$ as $\tilde{n}$:

$$\int_{\tilde{n}}^{\infty} Q(n) \left[1 - \omega(n) - \frac{T'(n)}{1 - T'(n)}\eta_n\right] f(n)dn - Q(\tilde{n}) \frac{T'(\tilde{n})}{1 - T'(\tilde{n})} Z_{\tilde{n}}^c z^*(\tilde{n}) f(\tilde{n}) \frac{1}{z^{*\prime}(\tilde{n})} = 0 \tag{30}$$

where $\omega(n) = \omega(z^*(n))$, $Q(n) = Q(z^*(n))$, $Q(\tilde{n}) = Q(z^*(\tilde{n})) = Q(\tilde{z})$, $T'(n) = T'(z^*(n))$ etc.[52] [53]

Next, following roughly the Appendix to Saez (2001), we define:

$$K(\tilde{n}) = -\int_{\tilde{n}}^{\infty} Q(n)\eta_n \frac{T'(n)}{1 - T'(n)} f(n)dn$$

$$D(\tilde{n}) = \frac{\eta_{\tilde{n}}}{Z_{\tilde{n}}^c z^*(\tilde{n})} z^{*\prime}(\tilde{n})$$

$$C(\tilde{n}) = \int_{\tilde{n}}^{\infty} Q(n)(1 - \omega(n)) f(n)dn$$

[51] Again we abuse notation by defining $\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*} \equiv 0$ if the tax schedule is non-differentiable at z^*.
[52] We break type $m's$ indifference and assume they locate at z^{*-} so that $z^*(m) = z^{*-}$.
[53] At all z^* where the tax schedule is differentiable we have $H(z^*(n)) = F(n)$ (by Lemma 1), which implies $h(z^*(n))z^{*\prime}(n) = f(n)$.

We can rewrite Equation 29 as follows:

$$K'(\tilde{n}) = D(\tilde{n})\left(K(\tilde{n}) + C(\tilde{n})\right)$$

Following the steps Saez (2001) uses to get to Equation (27) in his Appendix we get:

$$K(\tilde{n}) = -\int_{\tilde{n}}^{\infty} C'(n) \exp\left[-\int_{\tilde{n}}^{n} D(n')dn'\right] dn - C(\tilde{n})$$

Differentiating the above equation yields:

$$K'(\tilde{n}) = -\int_{\tilde{n}}^{\infty} C'(n) D(\tilde{n}) \exp\left[-\int_{\tilde{n}}^{n} D(n')dn'\right] dn$$

Plugging in the definitions for K', C', D, rearranging we get:

$$\frac{T'(\tilde{n})}{1-T'(\tilde{n})}\frac{Z_{\tilde{n}}^{c} z^{*}(\tilde{n})}{z^{*\prime}(\tilde{n})} f(\tilde{n}) = \int_{\tilde{n}}^{\infty}(1-\omega(n))\frac{Q(n)}{Q(\tilde{n})}\exp\left[\int_{\tilde{n}}^{n}\left(-\frac{\eta_{n'} z^{*\prime}(n')}{Z_{n'}^{c} z^{*}(n')}\right)dn'\right] f(n)dn \quad (31)$$

Mirrlees's (1971) first order condition for the optimal tax schedule is:

$$\left(1+\frac{u_l(\tilde{n})}{\tilde{n}u_c(\tilde{n})}\right)\frac{\tilde{n}^2 f(\tilde{n})}{-u_l(\tilde{n}) - l(\tilde{n})u_{ll}(\tilde{n}) + l(\tilde{n})u_l(\tilde{n})u_{cl}(\tilde{n})/u_c(\tilde{n})} =$$
$$\int_{\tilde{n}}^{\infty}\left(\frac{1}{u_c(n)} - \frac{W'(u(n))}{\lambda}\right)\exp\left[\int_{n}^{n'} -\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right] f(n)dn$$

where $l(n) = z^*(n)/n$ and $u(n) = u\left(z^*(n) - T\left(z^*(n)\right), z^*(n)/n\right)$ etc. We now show that Mirrlees's first order condition is mathematically equivalent to our Equation 31. Using the following definition:

$$SOC(z,n) = \frac{\partial}{\partial z}\left(u_c\left(z - T(z), \frac{z}{n}\right)(1-T'(z)) + \frac{1}{n}u_l\left(z - T(z), \frac{z}{n}\right)\right)$$

we can write $\frac{Z_n^c z^*(n)}{1-T'(n)} = \frac{-u_c(n)}{SOC(z^*(n),n)}$ and $\frac{1}{z^{*\prime}(n)} = \frac{-SOC(z^*(n),n)}{\frac{1}{n^2}(-u_l(n)-l(n)u_{ll}(n)+l(n)u_l(n)u_{cl}(n)/u_c(n))}$. Moreover, by $n's$ FOC we know: $T'(n) = \left(1+\frac{u_l(n)}{nu_c(n)}\right)$. Multiplying both sides of Mirrlees's formula by $u_c(\tilde{n})$ and using these identities we get

$$\frac{T'(\tilde{n})}{1-T'(\tilde{n})}\frac{Z_{\tilde{n}}^{c} z^{*}(\tilde{n})}{z^{*\prime}(\tilde{n})} f(\tilde{n}) = \int_{\tilde{n}}^{\infty}(1-\omega(n))\frac{u_c(\tilde{n})}{u_c(n)}\exp\left[\int_{\tilde{n}}^{n} -\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right] f(n)dn \quad (32)$$

Showing equivalence between Equation 31 and Equation 32 only requires showing that

$\frac{u_c(\tilde{n})}{u_c(n)}\exp\left[\int_{\tilde{n}}^{n}-\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]=\frac{Q(n)}{Q(\tilde{n})}\exp\left[\int_{\tilde{n}}^{n}-\frac{\eta_{n'}}{Z^c_{n'}}\frac{z^{*\prime}(n')}{z^*(n')}dn'\right]$. We have:

$$\begin{aligned}
&\frac{u_c(\tilde{n})}{u_c(n)}\frac{Q(\tilde{n})}{Q(n)}\exp\left[-\int_{\tilde{n}}^{n}\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\log\left(\frac{u_c(\tilde{n})}{u_c(n)}\frac{Q(\tilde{n})}{Q(n)}\right)-\int_{\tilde{n}}^{n}\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\left[\log\left(u_c(\tilde{n})Q(\tilde{n})\right)-\log\left(u_c(n)Q(n)\right)\right]-\int_{\tilde{n}}^{n}\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\int_{\tilde{n}}^{n}-\frac{d\log\left(u_c(n')Q(n')\right)}{dn'}-\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\int_{\tilde{n}}^{n}-\frac{d\left(u_c(n')Q(n')\right)}{dn'}\frac{1}{u_c(n')Q(n')}-\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\int_{\tilde{n}}^{n}-\frac{du_c(n')}{dn'}\frac{1}{u_c(n')}-\frac{l(n')u_{cl}(n')}{n'u_c(n')}dn'\right]\\
&=\exp\left[\int_{\tilde{n}}^{n}-\frac{u_{cc}(n')(1-T'(n'))+\frac{1}{n}u_{cl}(n')}{u_c(n')}z^{*\prime}(n')dn'\right]\\
&=\exp\left[\int_{\tilde{n}}^{n}-\frac{\eta_n}{Z^c_n}\frac{z^{*\prime}(n')}{z^*(n')}dn'\right]
\end{aligned}$$

where the third equality follows as $u_c(n)Q(n)$ is is an absolutely continuous function[54], the fifth equality follows as $\frac{du_c(n')Q(n')}{dn'}=Q(n')\frac{du_c(n')}{dn'}$ almost everywhere, and the final equality follows by the definition of the elasticities η_n and Z^c_n. Thus, our formula is equivalent to Mirrlees (1971) even if individuals have multiple optimal income levels.

A.9 Proof that $J_{1i}(\alpha)<0$

WLOG, assume that (m,α) is the only individual with multiple optima, and denote his multiple income levels as z^{*-} and z^{*+} (thus, we have dropped the i subscripts and no longer express the multiple income levels or the productivity level of this individual as functions of α). $J_1(\alpha)$ is given by:

$$\left(T(z^{*-})-T(z^{*+})\right)\frac{u_c^{*+}(\alpha)}{u_l^{*-}(\alpha)\frac{z^{*-}}{m^2}-u_l^{*+}(\alpha)\frac{z^{*+}}{m^2}}f(m|\alpha)\equiv J_1(\alpha)$$

where $u_c^{*+}(\alpha)=u_c\left(c(z^{*+}),\frac{z^{*+}}{m};\alpha\right)$ and $u_l^{*-}(\alpha)=u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)$ etc.

We know that $u_c^{*+}(\alpha)>0$ and we assume that $T(z^{*-})-T(z^{*+})<0$. Thus, to show $J_1(\alpha)<0$, we need to show $u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)z^{*-}-u_l\left(c(z^{*+}),\frac{z^{*+}}{m};\alpha\right)z^{*+}>0$. Suppose towards a contradiction that $u_l\left(c(z^{*+}),\frac{z^{*+}}{m};\alpha\right)z^{*+}\geq u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha\right)z^{*-}$.

[54] Continuity arises because $\lim_{n\to m^-}u_c(n)Q(n)=u_c^{*-}$ and $\lim_{n\to m^+}u_c(n)Q(n)=u_c^{*+}\frac{u_c^{*-}}{u_c^{*+}}=u_c^{*-}$. Absolute continuity follows due to our differentiability assumptions on $u(\cdot)$.

Consider the function $\hat{c}(z; m, \alpha, \bar{u})$ that implicitly solves $u\left(\hat{c}, \frac{z}{m}; \alpha\right) = \bar{u}$ where $\bar{u} = u\left(c(z^{*-}), \frac{z^{*-}}{m}; \alpha\right) = u\left(c(z^{*+}), \frac{z^{*+}}{m}; \alpha\right)$. Thus, $\hat{c}(z; m, \alpha, \bar{u})$ denotes the consumption level for type (m, α) such that for every income level z, utility remains constant at $\bar{u}$. By construction $\hat{c}(z^{*-}) = c(z^{*-})$ and $\hat{c}(z^{*+}) = c(z^{*+})$ (note, we will omit the arguments $m, \alpha, \bar{u}$ from $\hat{c}(\cdot)$ for ease of notation).

By the Mean Value Theorem we know that (omitting the α argument from the utility function):

$$z^{*+} u_l\left(\hat{c}(z^{*+}), \frac{z^{*+}}{m}\right) = z^{*-} u_l\left(\hat{c}(z^{*-}), \frac{z^{*-}}{m}\right) + (z^{*+} - z^{*-})\left(\frac{\partial(z u_l\left(\hat{c}(z), \frac{z}{m}\right))}{\partial z}\right)\Big|_{\tilde{z}}$$
$$= z^{*-} u_l\left(\hat{c}(z^{*-}), \frac{z^{*-}}{m}\right) + (z^{*+} - z^{*-})\left(u_l + \frac{z}{m} u_{ll} + z u_{cl} \hat{c}'(z)\right)\Big|_{\hat{c}(\tilde{z}), \tilde{z}}$$

for some $\tilde{z} \in (z^{*-}, z^{*+})$. By the fact that $u(\hat{c}(z), z/m)$ is constant for all z, we know: $\frac{\partial u(\hat{c}(z), \frac{z}{m})}{\partial z} = u_c\left(\hat{c}(z), \frac{z}{m}\right)\hat{c}'(z) + \frac{1}{m} u_l\left(\hat{c}(z), \frac{z}{m}\right) = 0$. Substituting in $\hat{c}'(z) = -\frac{u_l\left(\hat{c}(z), \frac{z}{m}\right)}{m u_c\left(\hat{c}(z), \frac{z}{m}\right)}$:

$$z^{*+} u_l\left(\hat{c}(z^{*+}), \frac{z^{*+}}{m}\right) = z^{*-} u_l\left(\hat{c}(z^{*-}), \frac{z^{*-}}{m}\right) + (z^{*+} - z^{*-})\left(u_l + \frac{z}{m} u_{ll} - \frac{z}{m} u_{cl} \frac{u_l}{u_c}\right)\Big|_{\hat{c}(\tilde{z}), \tilde{z}}$$

By our assumption and the fact that $\hat{c}(z^{*+}) = c(z^{*+})$ etc., we have $z^{*+} u_l\left(\hat{c}(z^{*+}), \frac{z^{*+}}{m}\right) \geq z^{*-} u_l\left(\hat{c}(z^{*-}), \frac{z^{*-}}{m}\right)$. Moreover, we know $z^{*+} - z^{*-} > 0$. Thus, it must be the case that $\left(u_l + \frac{z}{m} u_{ll} - \frac{z}{m} u_{cl} \frac{u_l}{u_c}\right)\Big|_{\hat{c}(\tilde{z}), \tilde{z}} \geq 0$. However, (SCP) yields the following $\forall n, c, z$:

$$-\frac{1}{n^2 u_c^2}\left(-u_c u_l - \frac{z}{n} u_{ll} u_c + \frac{z}{n} u_l u_{cl}\right) < 0$$

which simplifies to

$$\left(u_l + \frac{z}{n} u_{ll} - \frac{z}{n} \frac{u_l}{u_c} u_{cl}\right) < 0$$

A contradiction. Thus, $z^{*+} u_l\left(c(z^{*+}), \frac{z^{*+}}{m}\right) < z^{*-} u_l\left(c(z^{*-}), \frac{z^{*-}}{m}\right)$, meaning $J_1(\alpha) < 0$.

A.10 Proof of Proposition 3

Proof. To start, we assume that the tax schedule is differentiable at both optimal incomes z^{*-} and z^{*+} (we consider the non-differentiable case below in A.10.1). We begin by showing that if indifference curves are convex in (c, z) space, then if there exists an individual with multiple optima, the marginal tax rate must be lower at her highest optimal income relative to the marginal tax rate at her lowest optimal income: $T'(z^{*+}) < T'(z^{*-})$. Note, as we assume there is only one dimension of heterogeneity throughout this proof, we have dropped the α argument.

Suppose there exists an individual with productivity m that has multiple global optima z^{*-} and z^{*+} under the optimal tax schedule (i.e., jumping behavior exists under the optimal tax schedule).[55] Because we assume the optimal tax schedule is differentiable at z^{*-} and z^{*+}, we know that z^{*-}, z^{*+} must satisfy individual m's FOC:

$$(1 - T'(z^{*-}))u_c^{*-} = -\frac{1}{m}u_l^{*-}$$

and

$$(1 - T'(z^{*+}))u_c^{*+} = -\frac{1}{m}u_l^{*+}$$

where $u_c^{*-} = u_c\left(z^{*-} - T(z^{*-}), \frac{z^{*-}}{m}\right)$, $u_l^{*-} = u_l\left(z^{*-} - T(z^{*-}), \frac{z^{*-}}{m}\right)$ etc.

The indifference curve for individual n with utility $\bar{u}$ is the set of points $(\hat{c}(z; n, \bar{u}), z)$ in (c, z) space where $\hat{c}(z; n, \bar{u})$ implicitly solves $u(\hat{c}, z/n) = \bar{u}$. Implicitly differentiating this last condition w.r.t. z, the slope of this indifference curve at point $(\hat{c}(z; n, \bar{u}), z)$ is given by $\frac{-\frac{1}{n}u_l\left(\hat{c}(z;n,\bar{u}), \frac{z}{n}\right)}{u_c\left(\hat{c}(z;n,\bar{u}), \frac{z}{n}\right)}$. Given we assume indifference curves are convex, this amounts to assuming $\frac{\partial}{\partial z}\frac{-\frac{1}{n}u_l\left(\hat{c}(z;n,\bar{u}), \frac{z}{n}\right)}{u_c\left(\hat{c}(z;n,\bar{u}), \frac{z}{n}\right)} > 0 \ \forall \ n, \bar{u}$.

Next, we know that $(z^{*-} - T(z^{*-}), z^{*-})$ and $(z^{*+} - T(z^{*+}), z^{*+})$ lie on the same indifference curve for individual m (as they are two points in (c, z) space that give m the same level of utility). Because $z^{*-} < z^{*+}$, by our convexity assumption, we know that $\frac{-\frac{1}{m}u_l^{*-}}{u_c^{*-}} < \frac{-\frac{1}{m}u_l^{*+}}{u_c^{*+}}$. By individual m's FOC, this implies $1 - T'(z^{*-}) < 1 - T'(z^{*+})$ meaning $T'(z^{*+}) < T'(z^{*-})$. Thus, if indifference curves are convex in (c, z) space and there exists an individual with multiple optima, it must be the case that $T'(z^{*+}) < T'(z^{*-})$.

Now, consider perturbing the optimal tax schedule in the direction of the new function $\tau(z)$

$$\begin{cases} \tau(z) = 0 & \text{if } z \leq \tilde{z} - d\tilde{z}^2 \\ \tau(z) = z - \tilde{z} + d\tilde{z}^2 & \text{if } z \in [\tilde{z}, \tilde{z} + d\tilde{z}] \\ \tau(z) = d\tilde{z} & \text{if } z \in [\tilde{z} + d\tilde{z} + d\tilde{z}^2, z^{*-} + \delta_1] \\ \tau(z) = d\tilde{z}\frac{u_c^{*-}}{u_c^{*+}} & \text{if } z \geq z^{*-} + \delta_2 \end{cases}$$

where $\tilde{z} < z^{*-}$, and where $\delta_2 > \delta_1 > 0$ and $d\tilde{z} > 0$ are chosen s.t. $\tilde{z} + d\tilde{z} + d\tilde{z}^2 < z^{*-}$ and $\delta_2 < z^{*+} - z^{*-}$. We define $\tau(z)$ to change from $d\tilde{z}$ to $d\tilde{z}\frac{u_c^{*-}}{u_c^{*+}}$ in a twice continuously differentiable way between $z^{*-} + \delta_1$ and $z^{*-} + \delta_2$ yet remain under individual m's indifference curve. Similarly, we define $\tau(z)$ to change in a twice continuously differentiable way between $\tilde{z} - d\tilde{z}^2$ and $\tilde{z}$ and between $\tilde{z} + d\tilde{z}$ and $\tilde{z} + d\tilde{z} + d\tilde{z}^2$ (see Appendix A.4 for $\tau(z)$ defined between these two intervals). Note, with one dimension of heterogeneity, there

[55] The proof is easily extended to consider multiple individuals with multiple optima.

are no individuals that locate between z^{*-} and z^{*+}. Most importantly, under this perturbation, the jumping effect $J_2 = 0$ because $\tau(z^{*+}) = \tau(z^{*-})\frac{u_c^{*-}}{u_c^{*+}}$, thus the numerator in Equation 8 is 0, hence $\frac{\partial m}{\partial \mu}|_{\mu=0} = 0$ (i.e., the person with multiple optima doesn't change).

Using this perturbation, we can derive a condition that the optimal tax schedule must satisfy for $\tilde{z} < z^{*-}$:

$$\int_{\tilde{z}}^{z^{*-}} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* + \frac{u_c^{*-}}{u_c^{*+}} \int_{z^{*+}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* \\ - \frac{T'(\tilde{z})}{1-T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) = 0 \tag{33}$$

remembering that no one locates between (z^{*-}, z^{*+}).

Now consider the same $\tau(z)$ as in the main body of the text. We know that the optimal tax schedule for $\tilde{z} > z^{*+}$ must satisfy:

$$\int_{\tilde{z}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* - \frac{T'(\tilde{z})}{1-T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) = 0 \tag{34}$$

There will be no jumping effects from this perturbation as we assume there is just the one individual with multiple optima and we are perturbing the tax schedule above his top income: $\tilde{z} > z^{*+}$.

Taking the limit of Equation 33 as $\tilde{z} \to z^{*-}$ from the left, and taking the limit of Equation 34 as $\tilde{z} \to z^{*+}$ from the right, we know that the optimal tax schedule must satisfy the following two equations:

$$\lim_{(\tilde{z}\to z^{*-})^-} \left(\int_{\tilde{z}}^{z^{*-}} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* + \right. \\ \left. \frac{u_c^{*-}}{u_c^{*+}} \int_{z^{*+}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* - \frac{T'(\tilde{z})}{1-T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) \right) = 0$$

and

$$\lim_{(\tilde{z}\to z^{*+})^+} \left(\int_{\tilde{z}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* - \frac{T'(\tilde{z})}{1-T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) \right) = 0$$

We partially evaluate these limits to yield:

$$\frac{u_c^{*-}}{u_c^{*+}} \int_{z^{*+}}^{\infty} \left[1 - \omega(z^*) - \frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right] h(z^*)dz^* - \lim_{(\tilde{z}\to z^{*-})^-} \left(\frac{T'(\tilde{z})}{1-T'(\tilde{z})} Z_{\tilde{z}}^c \tilde{z} h(\tilde{z}) \right) = 0 \tag{35}$$

and

$$\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]h(z^*)dz^* - \lim_{(\tilde{z}\to z^{*+})^+}\left(\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})\right)=0 \quad (36)$$

Next, we will show that $\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) < \lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})$. First, we know the following is true for any individual with unique optimal income z^* (hence holds for any particular optimal income level $\tilde{z}$):

$$Z^c_{z^*}z^*h(z^*) = Z^c_{z^*}z^*f(n(z^*))\frac{\partial n(z^*)}{\partial z^*}$$

Implicitly differentiating the individual's FOC to obtain $\frac{\partial n(z^*)}{\partial z^*}$ and plugging into the above equation, we get:

$$Z^c_{z^*}z^*h(z^*) = \frac{-(1-T'(z^*))u^*_c f(n(z^*))}{\frac{z^*}{n^2}u^*_{cl}(1-T'(z^*))+\frac{1}{n^2}u^*_l+\frac{z^*}{n^3}u^*_{ll}}$$

Noting that $u^*_c(1-T'(z^*)) = -\frac{1}{n}u^*_l$ we get:[56]

$$\frac{Z^c_{z^*}z^*h(z^*)}{(1-T'(z^*))u^*_c} = \frac{f(n(z^*))n(z^*)^2}{\frac{z^*}{n(z^*)}u^*_{cl}\frac{u^*_l}{u^*_c}-u^*_l-\frac{z^*}{n(z^*)}u^*_{ll}} \quad (37)$$

By continuity of the density function $f(n)$ (which we assumed at the beginning of the paper), we get $\lim_{(z^*\to z^{*-})^-}f(n(z^*)) = f(m) = \lim_{(z^*\to z^{*+})^+}f(n(z^*))$. Moreover, by continuity of the utility function (and it's derivatives) as well as $T'(\cdot)$, we can take the limit through the utility functions and the tax schedule to yield:

$$\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) = \frac{T'(z^{*-})f(m)m^2}{\frac{z^{*-}}{m}u^{*-}_{cl}\frac{u^{*-}_l}{u^{*-}_c}-u^{*-}_l-\frac{z^{*-}}{m}u^{*-}_{ll}}$$

and

$$\lim_{(\tilde{z}\to z^{*+})^+}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) = \frac{T'(z^{*+})f(m)m^2}{\frac{z^{*+}}{m}u^{*+}_{cl}\frac{u^{*+}_l}{u^{*+}_c}-u^{*+}_l-\frac{z^{*+}}{m}u^{*+}_{ll}}$$

If $\frac{z}{n}u_{cl}\frac{u_l}{u_c}-u_l-\frac{z}{n}u_{ll}$ is increasing in z along each $n's$ indifference curve, then because z^{*+} and z^{*-} are both on the same indifference curve for individual m, we know (also using

[56]Note, the compensated elasticity need not be well-defined at z^{*-} and z^{*+} because the SOC may only hold weakly for multiple-optima individuals (see Lemma A.3). However, $\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})$ and $\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})$ are nonetheless well-defined as the RHS of Equation 37 is always well-defined by Appendix A.3.1 where we show $\frac{z}{n}u_{cl}\frac{u_l}{u_c}-u_l-\frac{z}{n}u_{ll}>0$.

$\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll} > 0$ by Appendix A.3.1):

$$\frac{f(m)m^2}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - u_l^{*-} - \frac{z^{*-}}{m}u_{ll}^{*-}} > \frac{f(m)m^2}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}} - u_l^{*+} - \frac{z^{*+}}{m}u_{ll}^{*+}}$$

And, given that $0 \leq T'(z^{*+}) < T'(z^{*-})$ we get:[57]

$$\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) > \lim_{(\tilde{z}\to z^{*+})^+}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\frac{1}{u_c(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})$$

which gives:

$$\lim_{(\tilde{z}\to z^{*-})^-}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}) > \frac{u_c^{*-}}{u_c^{*+}}\lim_{(\tilde{z}\to z^{*+})^+}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})$$

noting that we have passed the limit through the $\frac{1}{u_c(\tilde{z})}$ term, which is acceptable again by continuity.

Finally, by the above inequality and Equation 35, we have that:

$$\frac{u_c^{*-}}{u_c^{*+}}\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]h(z^*)dz^* - \frac{u_c^{*-}}{u_c^{*+}}\lim_{(\tilde{z}\to z^{*+})^+}\left(\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})\right) >$$
$$\frac{u_c^{*-}}{u_c^{*+}}\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]h(z^*)dz^* - \lim_{(\tilde{z}\to z^{*-})^-}\left(\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})\right) = 0$$

Dividing through by $\frac{u_c^{*-}}{u_c^{*+}}$ we get that Equation 36 cannot be satisfied:

$$\int_{z^{*+}}^{\infty}\left[1-\omega(z^*)-\frac{T'(z^*)}{1-T'(z^*)}\eta_{z^*}\right]h(z^*)dz^* - \lim_{(\tilde{z}\to z^{*+})^+}\left(\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})\right) > 0$$

Thus, if there exists an individual with multiple optimal incomes, it must be locally optimal to increase marginal tax rates at z^{*+} as long as tax rates are differentiable at z^{*-} and z^{*+}.

A.10.1 Proof of Proposition 3 when the tax schedule is non-differentiable

To complete the proof of Proposition 3, suppose that the tax schedule is not differentiable at z^{*-} and z^{*+} (if only one of $T(z^{*-})$ or $T(z^{*+})$ is non-differentiable the proof is essentially unchanged). First, note that marginal tax rates just above z^{*-} must be greater than marginal tax rates just below z^{*-}. Similarly, marginal tax rates just above z^{*+} must be greater than marginal tax rates just below z^{*+}. In other words, the two kink points must

[57]The fact that marginal tax rates are weakly positive follows from Proposition 3 in Mirrlees (1971).

be concave kink points in (c, z) space, otherwise our multiple optima individual would not find either z^{*-} or z^{*+} optimal (this is clear from an indifference curve diagram). Note that while $T(z^{*-})$ and $T(z^{*+})$ are defined in the limit by Equation 11, marginal tax rates between z^{*-} and z^{*+} are not defined by Equation 11 because no individuals locate between these income levels. Any tax schedule that connects $T(z^{*-})$ and $T(z^{*+})$ in between z^{*-} and z^{*+} that lies below type m's indifference curve (so that type m does not strictly prefer any of these income levels) yields the same total welfare. Hence, it suffices to show that we can find a welfare improving perturbation starting from a tax schedule such that type m is right indifferent at z^{*-} and left indifferent at z^{*+} as marginal tax rates are not pinned down uniquely between z^{*-} and z^{*+} (this therefore shows that any tax schedule with two optimal incomes, one or more of which is a kink point, cannot be optimal). Thus, let $T'^r(z^{*-})$ and $T'^l(z^{*+})$ satisfy the following two conditions:

$$(1 - T'^r(z^{*-}))u_c^{*-} = -\frac{1}{m}u_l^{*-}$$

and

$$(1 - T'^l(z^{*+}))u_c^{*+} = -\frac{1}{m}u_l^{*+}$$

where $T'^r(z^{*-})$ denotes the right derivative of the tax schedule at z^{*-}, $T'^l(z^{*+})$ denotes the left derivative of the tax schedule at z^{*+}, and $u_c^{*-} = u_c\left(z^{*-} - T(z^{*-}), \frac{z^{*-}}{m}\right)$ etc.[58] Now, consider perturbing the optimal tax schedule in the direction of a new function $\tau(z)$ so that the tax function is $T(z) + \mu\tau(z)$:

$$\begin{cases} \tau(z) = 0 & \text{if } z \leq z^{*-} \\ \tau(z) = (z^{*-} - z) & \text{if } z \in [z^{*-}, z^{*-} + \delta_1] \\ \tau(z) = (z - z^{*-} - 2\delta_1) & \text{if } z \in [z^{*-} + \delta_1, z^{*-} + 2\delta_1] \\ \tau(z) = 0 & \text{if } z \in [z^{*-} + 2\delta_1, z^{*+} - 2\delta_2] \\ \tau(z) = \gamma(z^{*+} - z - 2\delta_2) & \text{if } z \in [z^{*+} - 2\delta_2, z^{*+} - \delta_2] \\ \tau(z) = \gamma(z - z^{*+}) & \text{if } z \in [z^{*+} - \delta_2, z^{*+}] \\ \tau(z) = 0 & \text{if } z \geq z^{*+} \end{cases}$$

We are lowering marginal tax rates by a small amount just after the kink point at z^{*-} and raising marginal tax rates just before the second kink point at z^{*+}. Figure 9 illustrates the perturbation in blue dashed lines:

[58] We can, WLOG, consider a tax schedule with $T''^r(z^{*-})$ and $T''^l(z^{*+})$ sufficiently large, so that the SOC is strictly satisfied. Given indifference curves are convex, it is possible to have large $T''^r(z^{*-})$ and $T''^l(z^{*+})$ and still have a tax function that lies below the indifference curve of type m.

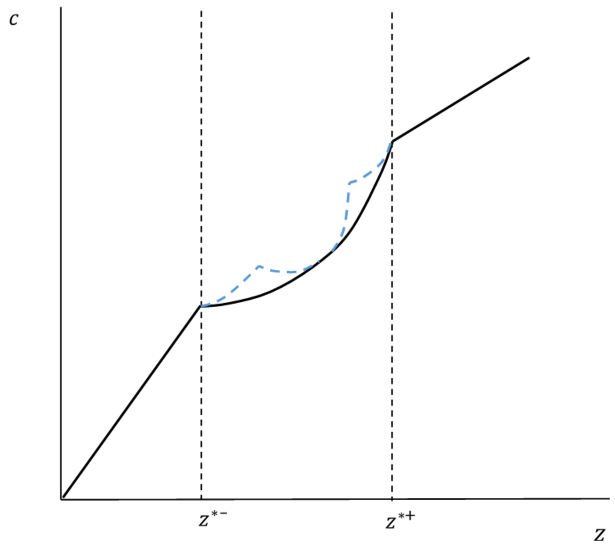

Figure 9: Perturbed tax schedule when there are kink points at z^{*-}, z^{*+}

where δ_1, δ_2 are chosen so that $z^{*-} + 2\delta_1 < z^{*+} - 2\delta_2$. We assume μ is sufficiently small so that there is no bunching at the two new kinks: $z^{*-} + \delta_1$ and $z^{*+} - \delta_2$ (we can do this because no one locates between (z^{*-}, z^{*+}) under T; so while a few people from the original kinks at z^{*-} and/or z^{*+} will "spread" out from this perturbation into the region (z^{*-}, z^{*+}), the extent of this "spreading" goes to 0 with μ). Moreover, we choose γ so that the individual with multiple optima isn't changing with the perturbation (this will become clearer later on in the proof):

$$\gamma = \sqrt{\frac{\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}}{\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}} \frac{u_c(z^{*-})}{u_c(z^{*+})}}$$

Now let us consider the effects of this perturbation. First, we will show that the first-order effect on the government's Lagrangian of perturbing the tax schedule in the direction of $\tau(z)$ will be 0 (i.e., $\partial\mathcal{L}/\partial\mu|_{\mu=0} = 0$). However, we will then show that the second-order effect of perturbing the tax schedule in the direction of $\tau(z)$ will be positive (i.e., $\partial^2\mathcal{L}/\partial\mu^2|_{\mu=0} > 0$), thus implying we were not at the optimal schedule. In order to derive the second variation of the tax perturbation in the direction of $\tau(z)$, we will need to write the effects of this perturbation (i.e., the mechanical, elastic, and jumping effects) in a more involved manner than in the text. In particular, we will write these effects as functions of μ so that we can then differentiate them w.r.t. μ (as opposed to simply evaluating the Gateaux derivatives at $\mu = 0$ as in the text). We now write each of these effects out, starting with the elastic effect. The elastic effect from this perturbation captures that individuals who initially bunch at the kinks now "spread out": individuals at z^{*-} spread to the right and individuals at z^{*+} spread to the left. The impact that each spreading individual has on the Lagrangian is equal to $\lambda\frac{\partial T(z(n))}{\partial\mu}$. The next question is how many individuals spread out from the original kinks? We need to determine how the individual whose FOC is satisfied under the right marginal tax rate at z^{*-} changes with μ; let us denote this individual as a function of μ by $n^r(\mu)$. Then, we can write the elastic effect of this perturbation as $\lambda\int_{n^r}^{m}\frac{\partial T(z(n))}{\partial\mu}f(n)dn$. We can determine $\frac{\partial n^r}{\partial\mu}$ using the implicit function theorem and the FOC for type $n^r(\mu)$:[59]

$$
\begin{aligned}
&u_c\left(z^{*-}-T(z^{*-})-\mu\tau(z^{*-}),\frac{z^{*-}}{n^r}\right)\left(1-T'^r(z^{*-})-\mu\tau'^r(z^{*-})\right)\\
&+\frac{1}{n^r}u_l\left(z^{*-}-T(z^{*-})-\mu\tau(z^{*-}),\frac{z^{*-}}{n^r}\right)=0
\end{aligned}
\tag{38}
$$

Noting $n^r(0)=m$ and $\frac{\partial n^r}{\partial\mu}|_{\mu=0}=m^2\frac{u_c^{*-}\tau'^r(z^{*-})}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}}-u_l^{*-}-\frac{z^{*-}}{m}u_{ll}^{*-}}<0$ (as $\tau'^r(z^{*-})=-1$ and the denominator is positive by Appendix A.3.1). An analogous reasoning works at the second kink z^{*+}, defining $n^l(\mu)$ as the individual whose FOC is satisfied under the left marginal tax rate z^{*+} with $\frac{\partial n^l}{\partial\mu}|_{\mu=0}=m^2\frac{u_c^{*+}\tau'^l(z^{*+})}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}}-u_l^{*+}-\frac{z^{*+}}{m}u_{ll}^{*+}}>0$ (as $\tau'^l(z^{*+})=\gamma>0$ and the denominator is positive, again by Appendix A.3.1). So the elastic effect is:

$$
\begin{aligned}
&\lambda\int_{n^r}^{m}\frac{\partial T(z(n))}{\partial\mu}f(n)dn+\lambda\int_{m}^{n^l}\frac{\partial T(z(n))}{\partial\mu}f(n)dn\\
&=\lambda\int_{n^r}^{m}T'^r(z(n))\frac{\partial z(n)}{\partial\mu}f(n)dn+\lambda\int_{m}^{n^l}T'^l(z(n))\frac{\partial z(n)}{\partial\mu}f(n)dn
\end{aligned}
$$

There is also a mechanical effect. Recall that the mechanical effect of the tax schedule holds behavioral responses constant (i.e., holds $z(n)$ constant). Also, everyone outside of the range $[z^{*-},z^{*+}]$ is unaffected by the perturbation:

$$
\begin{aligned}
&\int_{n^r}^{m}\frac{\partial\left[W(u(n))+\lambda\mu\tau(z(n))\right]}{\partial\mu}\Big|_{z(n)}f(n)dn+\int_{m}^{n^l}\frac{\partial\left[W(u(n))+\lambda\mu\tau(z(n))\right]}{\partial\mu}\Big|_{z(n)}f(n)dn=\\
&\int_{n^r}^{m}\left[W_u(u(n))u_c(n)\tau(z(n))+\lambda\tau(z(n))\right]f(n)dn+\int_{m}^{n^l}\left[W_u(u(n))u_c(n)\tau(z(n))+\lambda\tau(z(n))\right]f(n)dn
\end{aligned}
$$

where $u(n)$ is shorthand notation for $u\left(z(n)-T\left(z(n)\right)-\mu\tau\left(z(n)\right),\frac{z(n)}{n}\right)$.

Finally, there is a jumping effect induced by this perturbation given by:

$$
\begin{aligned}
&-\lambda[T\left(z^{*-}(m(\mu),\mu)\right)-T\left(z^{*+}(m(\mu),\mu)\right)]\times\\
&\qquad\frac{u_c^{*-}(\mu)\tau\left(z^{*-}(m(\mu),\mu)\right)-u_c^{*+}(\mu)\tau\left(z^{*+}(m(\mu),\mu)\right)}{u_l^{*-}(\mu)\frac{z^{*-}(m(\mu),\mu)}{m(\mu)^2}-u_l^{*+}(\mu)\frac{z^{*+}(m(\mu),\mu)}{m(\mu)^2}}f\left(m(\mu)\right)
\end{aligned}
$$

[59]Note z^{*-} is not a function of μ as $n^r(\mu)$ is defined as the person who is right indifferent at z^{*-}.

where $u_c^{*-}(\mu) = u_c\left(z^{*-}\left(m(\mu),\mu\right) - T\left(z^{*-}(m(\mu),\mu)\right) - \mu\tau\left(z^{*-}(m(\mu),\mu)\right), \frac{z^{*-}(m(\mu),\mu)}{m(\mu)}\right)$ etc.; $m(\mu)$ captures the productivity level of the individual with multiple optima under the perturbed scheduled while $(z^{*-}(m(\mu),\mu), z^{*+}(m(\mu),\mu))$ capture his lower and upper optimal incomes under the perturbed schedule. Note, it is important to differentiate between m and $m(\mu)$: m is the productivity level of the individual with multiple optima under the non-perturbed tax schedule: $m = m(\mu)|_{\mu=0}$. Similarly, (z^{*-}, z^{*+}) are the lower and upper optimal income levels chosen by m under the non-perturbed tax schedule: $z^{*-} = z^{*-}(m(\mu),\mu)|_{\mu=0}$ and $z^{*+} = z^{*+}(m(\mu),\mu)|_{\mu=0}$.

Because $\tau(z^{*-}) = \tau(z^{*+}) = 0$, both the jumping effect and the mechanical effect of this perturbation are 0 when evaluated at $\mu = 0$. Moreover, the elastic effect is also 0 when evaluated $\mu = 0$ (as the upper and lower limits of integration in the two integrals in the elastic effect are identical in the limit as $\mu \to 0$). Thus, the three effects of the perturbation are all 0. However, we know that for a tax schedule to be a maximal schedule, the second variation must be negative (if it's positive, this means that the tax schedule is actually a locally minimal tax schedule). Taking the derivative of the elastic effect w.r.t. μ and evaluating at $\mu = 0$ yields, recognizing that $\left(\frac{\partial z(n)}{\partial \mu}\Big|_{n=n^r}\right)\Big|_{\mu=0} = \frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}$ and $\left(\frac{\partial z(n)}{\partial \mu}\Big|_{n=n^l}\right)\Big|_{\mu=0} = -\gamma\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}$.[60]

$$-\lambda\frac{\partial n^r}{\partial \mu}\Big|_{\mu=0} T'^r(z^{*-})\frac{Z^c_{z^{*-}} z^{*-}}{(1-T'^r(z^{*-}))} f(m) - \lambda\frac{\partial n^l}{\partial \mu}\Big|_{\mu=0} T'^l(z^{*+})\gamma\frac{Z^c_{z^{*+}} z^{*+}}{(1-T'^l(z^{*+}))} f(m)$$

Note, when differentiating the elastic effect w.r.t. μ and evaluating at $\mu = 0$, we can ignore the derivatives of the two integrands w.r.t. μ as the intervals over which they are integrated $\to 0$ as $\mu \to 0$. The derivative of the mechanical effect w.r.t. μ evaluated at $\mu = 0$ is 0 given that $(\tau(z(n))|_{n=n^r})\,|_{\mu=0} = \tau(z^{*-}) = 0$ and $(\tau(z(n))|_{n=n^l})\,|_{\mu=0} = \tau(z^{*+}) = 0$. And the derivative of the jumping effect evaluated at $\mu = 0$ is:[61]

$$-\lambda[T\left(z^{*-}\right) - T\left(z^{*+}\right)]\times \frac{u_c^{*-}\tau'^r(z^{*-})\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0} - u_c^{*+}\tau'^l(z^{*+})\frac{\partial z^{*+}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0}}{u_l^{*-}\frac{z^{*-}}{m^2} - u_l^{*+}\frac{z^{*+}}{m^2}}$$

where $u_c^{*-} = u_c\left(z^{*-} - T(z^{*-}), \frac{z^{*-}}{m}\right)$ etc. Note, however, that we have chosen $\tau(z)$

[60]These formulas are derived, for example, by substituting in $n = n^r(\mu)$ into the expression for $\frac{\partial z(n)}{\partial \mu}$ in Equation 2, and then evaluating at $\mu = 0$.

[61]Note, while $m(\mu)$, $z^{*-}(m(\mu)\mu)$ and $z^{*+}(m(\mu)\mu)$ are all functions of μ, the derivatives of these terms are all multiplied by $\tau(z^{*-}) = \tau(z^{*+}) = 0$. Moreover, the derivative of $\tau\left(z^{*-}(m(\mu),\mu)\right)$ with respect to μ (at $\mu = 0$) is $\tau'^r(z^{*-})\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}|_{\mu=0}$ because the income for which an individual is indifferent, $z^{*-}(m(\mu),\mu)$, is increasing with μ. By similar logic, the derivative of $\tau\left(z^{*-}(m(\mu),\mu)\right)$ with respect to μ (at $\mu = 0$) is $\tau'^l(z^{*+})\frac{\partial z^{*+}(m(\mu),\mu)}{\partial \mu}|_{\mu=0}$.

such that $\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0} = \frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}$ and $\frac{\partial z^{*+}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0} = -\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}\gamma$.[62] Given that $\tau'^r(z^{*-}) = -1$ and $\tau'^l(z^{*+}) = \gamma$:

$$u_c^{*-}\tau'^r(z^{*-})\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0} - u_c^{*+}\tau'^l(z^{*+})\frac{\partial z^{*+}(m(\mu),\mu)}{\partial \mu}\Big|_{\mu=0} =$$
$$- u_c^{*-}\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})} + u_c^{*+}\gamma^2\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})} = 0$$

where the last equality follows given our definition of γ. Thus, the derivative of the jumping effect w.r.t. μ evaluated at $\mu = 0$ is also 0. Thus, for the second variation in the direction of $\tau(z)$ to be negative we require the derivative of the elasticity effect to be negative:

$$-\lambda\frac{\partial n^r}{\partial \mu}\Big|_{\mu=0} T'^r(z^{*-})\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}f(m) - \lambda\frac{\partial n^l}{\partial \mu}\Big|_{\mu=0} T'^l(z^{*+})\gamma\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}f(m) < 0$$

We can plug in the definition of $\frac{\partial n^r}{\partial \mu}|_{\mu=0}$ and $\frac{\partial n^l}{\partial \mu}|_{\mu=0}$ (and divide through by m^2) to yield:

$$- \lambda\frac{u_c^{*-}\tau'^r(z^{*-})}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - u_l^{*-} - \frac{z^{*-}}{m}u_{ll}^{*-}}T'^r(z^{*-})\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}f(m) -$$
$$\lambda\frac{u_c^{*+}\tau'^l(z^{*+})}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}} - u_l^{*+} - \frac{z^{*+}}{m}u_{ll}^{*+}}T'^l(z^{*+})\gamma\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}f(m) < 0$$

Noting that λ is the positive Lagrange multiplier on the budget constraint, $\tau'^r(z^{*-}) = -1$, and $\tau'^l(z^{*+}) = \gamma$:

$$\frac{u_c^{*-}}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - u_l^{*-} - \frac{z^{*-}}{m}u_{ll}^{*-}}T'^r(z^{*-})\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}f(m) -$$
$$\frac{u_c^{*+}}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}} - u_l^{*+} - \frac{z^{*+}}{m}u_{ll}^{*+}}T'^l(z^{*+})\gamma^2\frac{Z^c_{z^{*+}} z^{*+}}{1-T'^l(z^{*+})}f(m) < 0$$

Substituting in our definition of γ and dividing through by $u_c^{*-}\frac{Z^c_{z^{*-}} z^{*-}}{1-T'^r(z^{*-})}$ we get:

$$\frac{T'^r(z^{*-})f(m)}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - u_l^{*-} - \frac{z^{*-}}{m}u_{ll}^{*-}} - \frac{T'^l(z^{*+})f(m)}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}} - u_l^{*+} - \frac{z^{*+}}{m}u_{ll}^{*+}} < 0 \tag{39}$$

[62]Note, $\frac{\partial m(\mu)}{\partial \mu}|_{\mu=0} = 0$ as $\tau(z^{*-}) = \tau(z^{*+})$. Hence $\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}|_{\mu=0} = \left(\frac{\partial z^{*-}(m(\mu),\mu)}{\partial \mu}|_{m(\mu)}\right)|_{\mu=0}$.

But we showed previously that:

$$\frac{T'^{r}(z^{*-})f(m)}{\frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - u_l^{*-} - \frac{z^{*-}}{m}u_{ll}^{*-}} - \frac{T'^{l}(z^{*+})f(m)}{\frac{z^{*+}}{m}u_{cl}^{*+}\frac{u_l^{*+}}{u_c^{*+}} - u_l^{*+} - \frac{z^{*+}}{m}u_{ll}^{*+}} > 0$$

given our assumptions on convexity of indifference curves and $\frac{z}{n}u_{cl}\frac{u_l}{u_c} - u_l - \frac{z}{n}u_{ll}$ is increasing in z along each n's indifference curve. Hence, the second variation must actually be strictly positive. Thus, we know that our tax schedule is a local minimizer of welfare; hence, it cannot ever be locally optimal to have an individual with two optimal incomes at a kink point. Hence, with one dimension of heterogeneity, it can never be optimal to have an individual with multiple optimal incomes.

□

A.11 Proof to Corollary 2

Proof. First, we will show that those with the same $x = n\alpha^{\frac{1}{1+k}}$ will pick the same optimal income level for any tax schedule:

$$\begin{aligned} z^*(n,\alpha) &= \underset{z}{\text{argmax}}\ \alpha v(z - T(z)) - \frac{\left(\frac{z}{n}\right)^{1+k}}{1+k} \\ &= \underset{z}{\text{argmax}}\ v(z - T(z)) - \frac{\left(\frac{z}{n\alpha^{\frac{1}{1+k}}}\right)^{1+k}}{1+k} \\ &= \underset{z}{\text{argmax}}\ v(z - T(z)) - \frac{\left(\frac{z}{x}\right)^{1+k}}{1+k} = z^*(x) \end{aligned}$$

Thus, $z^*(n,\alpha) = z^*(x)$, i.e., those with the same x choose the same optimal income. We will now show that the government problem collapses to a one-dimension problem. The government problem is given by:

$$\begin{aligned} &\max_{T(z)} \int_A \int_N W\left(\alpha v(c^*(n,\alpha)) - \frac{\left(\frac{z^*(n,\alpha)}{n}\right)^{1+k}}{1+k}; n, \alpha\right) f(n|\alpha)dndF(\alpha) \\ &\text{s.t.} \int_A \int_N c^*(n,\alpha)f(n|\alpha)dndF(\alpha) + E \le \int_A \int_N z^*(n,\alpha)f(n|\alpha)dndF(\alpha) \end{aligned}$$

where $c^*(n,\alpha) = z^*(n,\alpha) - T(z^*(n,\alpha))$. Because x is increasing in n, we can do the following change of variables:

$$\max_{T(z)} \int_A \int_X W\left(\alpha v(c^*(x)) - \frac{\alpha\left(\frac{z^*(x)}{x}\right)^{1+k}}{1+k}; n(x,\alpha), \alpha\right) g(x|\alpha)dxdF(\alpha)$$

$$\text{s.t.} \int_A \int_X c^*(x)g(x|\alpha)dxdF(\alpha) + E \leq \int_A \int_X z^*(x)g(x|\alpha)dxdF(\alpha)$$

where $g(x|\alpha) = f(n(x,\alpha)|\alpha)\frac{\partial n(x,\alpha)}{\partial x} = f(n(x,\alpha)|\alpha)\alpha^{\frac{-1}{1+k}}$. Denoting $\widetilde{W}(u^*(x); x) = \int_A W(\alpha u^*(x); n(x,\alpha), \alpha)dF(\alpha|x)$ (where $u^*(x) = v(c^*(x)) - \frac{\left(\frac{z^*(x)}{x}\right)^{1+k}}{1+k}$) and switching the order of integration we get:

$$\max_{T(z)} \int_X \widetilde{W}\left(v(c^*(x)) - \frac{\left(\frac{z^*(x)}{x}\right)^{1+k}}{1+k}; x\right) g(x)dx$$

$$\text{s.t.} \int_X c^*(x)g(x)dx + E \leq \int_X z^*(x)g(x)dx$$

Thus, the government's problem collapses to a one-dimension problem with new social welfare function $\widetilde{W}(u;x)$. Note, that because we assume $f(n)$ is continuous, $g(x)$ is also continuous (the change of variables above shows this). Therefore, we can apply Corollary 1.

□

A.12 Proof of Proposition 4

Proof. Denote the two utility functions: $u\left(c, \frac{z}{n}; \alpha_1\right) = u^{(1)}\left(c, \frac{z}{n}\right)$ and $u\left(c, \frac{z}{n}; \alpha_2\right) = u^{(2)}\left(c, \frac{z}{n}\right)$. Let p denote the proportion of individuals who are type 1. To start with, consider $p = 0$, so that we have unidimensional heterogeneity. Define T_0 as the (assumed unique) optimal tax schedule when $p = 0$. To prove Proposition 4, we will proceed as follows: first, we will show that when the proportion of type 1 individuals is p, the optimal tax schedule, T_p, tends to T_0 as p tends to 0. This will imply that when p is very small but strictly positive, the optimal tax schedule $T_p \approx T_0$. Second, we will show that under T_0, we can find a $u^{(1)}$ with sufficiently flat indifference curves so that there exists a type 1 individual with multiple optimal income levels under T_0. Finally, using the fact that $T_p \approx T_0$ for very small but positive p, we will note that we can construct an example where both types are present and there exists a type 1 individual with multiple optimal incomes.

First, we show that if we allow p percent of individuals to be type 1, $T_p \to T_0$ as

$p \to 0$, where T_p denotes the optimal tax schedule when p percent of individuals are type 1. Suppose not, so that $T_p \to S \neq T_0$. Denote total welfare of type 2 under T_0 as $U^{(2),T_0}$ and total welfare of type 1 under T_0 as $U^{(1),T_0}$. Denote total welfare under S for the two types as $U^{(1),S}$ and $U^{(2),S}$. Note that $U^{(2),S} < U^{(2),T_0}$ as T_0 is the unique welfare maximizing tax schedule for type 2 individuals. For very small p, T_p is arbitrarily close to S, so that total utility (which is continuous in the tax schedule) is arbitrarily close to to $pU^{(1),S} + (1-p)U^{(2),S}$. But for small enough p we know that $pU^{(1),S} + (1-p)U^{(2),S} < pU^{(1),T_0} + (1-p)U^{(2),T_0}$ as $U^{(2),S} < U^{(2),T_0}$. Hence, for all sufficiently small p, T_0 yields higher utility than S, which is a contradiction. Hence $T_p \to T_0$ as $p \to 0$.

Now, let's specify the functional forms for utility: $u^{(i)} = c - \frac{\left(\frac{z}{n}\right)^{1+\alpha_i}}{1+\alpha_i}$ for $i = \{1, 2\}$, where we choose $0 < \alpha_1 < \alpha_2$ so that type 1 has flatter indifference curves. It is easy to check that these utility functions satisfy the conditions specified in Proposition 3, and that they satisfy the (SCP). Next we select the productivity support for type 2 to satisfy $\underline{n}^{(2)} > 0$ and $\overline{n}^{(2)} < 1$ where $\underline{n}^{(i)}$ and $\overline{n}^{(i)}$ denote the minimum and maximum productivity values that type i can have, respectively. Next, we choose the productivity support for type 1 such that $\underline{n}^{(1)} = \left(\underline{n}^{(2)\frac{\alpha_2+1}{\alpha_2}}\right)^{\frac{\alpha_1}{\alpha_1+1}}$ and $\overline{n}^{(1)} = \left(\overline{n}^{(2)\frac{\alpha_2+1}{\alpha_2}}\right)^{\frac{\alpha_1}{\alpha_1+1}}$. This productivity range for type 1 ensures that both type 1 and type 2 will locate at the minimum and maximum incomes chosen in society.[63] Moreover, because $\underline{n}^{(2)} > 0$ and $\overline{n}^{(2)} < 1$, we also know that $\underline{n}^{(1)} > 0$ and $\overline{n}^{(1)} < 1$.

Now, consider the optimal tax schedule when $p = 0$, T_0, and consider a type 2 individual $(\tilde{n}^{(2)}, \alpha_2)$ with unique global optimal income $z^*(\tilde{n}^{(2)}, \alpha_2) \equiv \tilde{z}$, for which $T_0(z)$ is twice continuously differentiable at $\tilde{z}$ and $T_0''(\tilde{z}) < 0$.[64] [65] Note that $\frac{z^*(n,\alpha_i)}{n} < 1$ whenever $T_0'(z^*(n, \alpha_i))$ exists. This is because whenever the tax schedule is differentiable, $z^*(n, \alpha_i)$ solves (by agent's FOC):

$$\frac{z^*}{n} = n^{\frac{1}{\alpha_i}} \left(1 - T_0'(z^*)\right)^{\frac{1}{\alpha_i}}$$

and because $0 < n < 1$, $T_0' \in [0, 1)$, $\alpha_i > 0$ for $i \in 1, 2$, implying that $0 < z^*(n, \alpha_i)/n < 1$ (note, Mirrlees (1971) showed $T_0' \in [0, 1)$ with one dimension of heterogeneity). Thus, $0 < \tilde{z}/\tilde{n}^{(2)} < 1$.

Next, we know that $1 - T'(\tilde{z}) - \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_2} \frac{1}{\tilde{n}^{(2)}} = 0$ (as $\tilde{z}$ satisfies $(\tilde{n}^{(2)}, \alpha_2)'s$ FOC). We also know that $\left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_2} < \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_1}$ as $\alpha_2 > \alpha_1 > 0$ and $\left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right) < 1$. Thus, we know that $1 - T'(\tilde{z}) - \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_1} \frac{1}{\tilde{n}^{(2)}} < 0$, i.e., $z^*(\tilde{n}^{(2)}, \alpha_1) \neq \tilde{z}$. In fact, because $1 - T'(\tilde{z}) - \left(\frac{\tilde{z}}{n}\right)^{\alpha_1} \frac{1}{n}$ is increasing in n, we know that $\tilde{n}^{(1)} > \tilde{n}^{(2)}$ where $\tilde{n}^{(1)}$ is the productivity level for type 1

[63] We have used the fact that marginal rates are 0 at the bottom and the top of the income distribution by Propositions 6 and 7.

[64] Each type 2 individual will have a unique global optimal under T_0 by Proposition 3.

[65] With a bounded skill distribution, all optimal tax schedules will have some decreasing component as optimal marginal tax rates are 0 at the top and bottom of the income distribution, see Propositions 6 and 7.

that solves: $1 - T'(\tilde{z}) - \left(\frac{\tilde{z}}{\tilde{n}^{(1)}}\right)^{\alpha_1} \frac{1}{\tilde{n}^{(1)}} = 0$.

Next, consider the SOC of type $(\tilde{n}^{(2)}, \alpha_2)$ at $\tilde{z}$. Given $\tilde{z}$ is unique for type $(\tilde{n}^{(2)}, \alpha_2)$, then by agent maximization, $SOC(\tilde{z}; \tilde{n}^{(2)}, \alpha_2) < 0$:[66]

$$SOC(\tilde{z}; \tilde{n}^{(2)}, \alpha_2) = -\frac{\alpha_2}{(\tilde{n}^{(2)})^2} \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_2 - 1} - T_0''(\tilde{z}) < 0$$

Now consider the SOC for the individual $(\tilde{n}^{(1)}, \alpha_1)$ who locates at $\tilde{z}$:

$$SOC(\tilde{z}; \tilde{n}^{(1)}, \alpha_1) = -\frac{\alpha_1}{(\tilde{n}^{(1)})^2} \left(\frac{\tilde{z}}{\tilde{n}^{(1)}}\right)^{\alpha_1 - 1} - T_0''(\tilde{z})$$

Because $\tilde{n}^{(1)} > \tilde{n}^{(2)}$ we know that

$$SOC(\tilde{z}; \tilde{n}^{(1)}, \alpha_1) > -\frac{\alpha_1}{(\tilde{n}^{(2)})^2} \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_1 - 1} - T_0''(\tilde{z})$$

Because $-T_0''(\tilde{z}) > 0$ we can find a small but strictly positive α_1 such that $SOC(\tilde{z}; \tilde{n}^{(1)}, \alpha_1) > 0$, e.g., any $\alpha_1 < -T_0''(\tilde{z})\tilde{n}^{(2)}\tilde{z}$ will make the SOC positive.[67] But if the SOC is positive, this implies that this is, in fact, a local minimum for $(\tilde{n}^{(1)}, \alpha_1)$, so that the type 1 individual whose FOC is satisfied at $\tilde{z}$ (this individual is unique by SCP) does not locate at $\tilde{z}$. Hence, no type 1 individual locates at $\tilde{z}$.

Finally, for sufficiently small $p > 0$, we know that $T_p''(\tilde{z}) \approx T_0''(\tilde{z})$, so that the SOC is still positive for this type 1 individual whose FOC is satisfied at $\tilde{z}$; thus no type 1 individual locates at $\tilde{z}$. Moreover, by our choice of the support for type 1 individuals, we know that some type 1 individual chooses to locate at the minimum income level chosen in society and another type 1 individual chooses to locate at the maximum income level chosen in society for any p.[68] Putting these facts together implies that there must be a jump discontinuity in the optimal income function $z^*(n, \alpha_1)$, which in turn implies there must be a type 1 individual with two global optimal income levels. Hence, for sufficiently small p, we know that there must exist an individual with multiple optimal incomes.

□

[66] See Lemma 3.

[67] If $\alpha_1 < -T_0''(\tilde{z})\tilde{n}^{(2)}\tilde{z}$ we get $SOC(\tilde{z}; \tilde{n}^{(1)}, \alpha_1) > T_0''(\tilde{z}) \left(\frac{\tilde{z}}{\tilde{n}^{(2)}}\right)^{\alpha_1} - T_0''(\tilde{z})$. Because $\frac{\tilde{z}}{\tilde{n}^{(2)}} < 1$ and $-T_0''(\tilde{z}) > 0$, we get $SOC(\tilde{z}; \tilde{n}^{(1)}, \alpha_1) > 0$.

[68] Propositions 6 and 7 show that bottom and top tax rates are 0 for any p.

A.13 Proof of Proposition 5

Proof. Suppose there are two types: α_1 and α_2; the proof is easily extended to more types. Suppose there exists an individual of type α_1 that has multiple optimal incomes, $z^{*-}(\alpha_1), z^{*+}(\alpha_1)$. We know Equation 11 holds $\forall$ z in some neighborhoods around $z^{*-}(\alpha_1)$ and $z^{*+}(\alpha_1)$ by Assumption 3 and the fact that $T(z)$ is twice continuously differentiable at all but a finite set of points. Thus, we can take the limit of Equation 11 as $\tilde{z} \to z^{*-}$ from the left and subtract off the limit of Equation 11 as $\tilde{z} \to z^{*-}$ from the right to get:

$$\lim_{(\tilde{z}\to z^{*-})^+}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})+J_1(\alpha_1)p(\alpha_1)\right)=\lim_{(\tilde{z}\to z^{*-})^-}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z})+J_2(\alpha_1)p(\alpha_1)\right)$$

Let us suppose that the tax schedule is differentiable at z^{*-}. We can simplify this equation (using the fact that the elasticity effect for type α_2 is continuous at z^{*-} because we assume the tax schedule is differentiable at z^{*-}):

$$J_1(\alpha_1)p(\alpha_1)-J_2(\alpha_1)p(\alpha_1)=\lim_{(\tilde{z}\to z^{*-})^-}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z},\alpha_1}\tilde{z}h(\tilde{z}|\alpha_1)p(\alpha_1)\right)$$

as $\lim_{(\tilde{z}\to z^{*-})^-}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}|\alpha_2)p(\alpha_2)\right)=\lim_{(\tilde{z}\to z^{*-})^+}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}|\alpha_2)p(\alpha_2)\right)$ given continuity in the elasticity effect for type α_2, and $\lim_{(\tilde{z}\to z^{*-})^+}\left(-\frac{T'(\tilde{z})}{1-T'(\tilde{z})}Z^c_{\tilde{z}}\tilde{z}h(\tilde{z}|\alpha_1)\right)p(\alpha_1)=0$ given no α_1 type locates just to the right of z^{*-}.

Substituting in the expressions for J_1, J_2, Z^c, and using the following identity that holds at all $\tilde{z}$ for which the tax schedule is twice differentiable:

$$\begin{aligned}h(\tilde{z}|\alpha)&=f(n(\tilde{z},\alpha)|\alpha)\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}}\\&=f(n(\tilde{z},\alpha)|\alpha)\left(\frac{u^*_{cc}(1-T'(z^*))^2+\frac{2}{n(z^*,\alpha)}u^*_{cl}(1-T'(z^*))+\frac{1}{n(z^*,\alpha)^2}u^*_{ll}-u^*_cT''(z^*)}{\frac{z^*}{n(z^*,\alpha)^2}u^*_{cl}(1-T'(z^*))+\frac{1}{n(z^*,\alpha)^2}u^*_l+\frac{z^*}{n(z^*,\alpha)^3}u^*_{ll}}\right)\Bigg|_{z^*=\tilde{z}}\end{aligned}$$

we get:

$$\begin{aligned}&\left(T(z^{*-})-T(z^{*+})\right)\frac{u^{*-}_c}{\frac{1}{m^2}\left(u^{*-}_l z^{*-}-u^{*+}_l z^{*+}\right)}f(m,\alpha_1)=\\&\lim_{(\tilde{z}\to z^{*-})^-}-\frac{T'(\tilde{z})\tilde{u}_c}{\frac{\tilde{z}}{n(\tilde{z},\alpha_1)}\tilde{u}_{cl}\frac{\tilde{u}_l}{\tilde{u}_c}-\tilde{u}_l-\frac{\tilde{z}}{n(\tilde{z},\alpha_1)}\tilde{u}_{ll}}n(\tilde{z},\alpha_1)^2f(n(\tilde{z},\alpha_1),\alpha_1)\end{aligned}\tag{40}$$

where $u^{*-}=u\left(z^{*-}-T(z^{*-}),\frac{z^{*-}}{m};\alpha_1\right)$, $\tilde{u}=u\left(\tilde{z}-T(\tilde{z}),\frac{\tilde{z}}{n(\tilde{z},\alpha_1)};\alpha_1\right)$ etc., and where we have substituted $(1-T'(z^*))=-\frac{\frac{1}{n(z^*,\alpha_1)}u^*_l}{u^*_c}$, and where $m=n(z^{*-},\alpha_1)$ denotes the productivity level of the individual with multiple optima.

We will now use the Mean Value Theorem to rewrite the LHS of Equation 40. Consider the function $\hat{c}(z; m, \alpha_1, \bar{u})$ which implicitly solves $u\left(\hat{c}, \frac{z}{m}; \alpha_1\right) = \bar{u}$, where $\bar{u} = u\left(z^{*-} - T(z^{*-}), \frac{z^{*-}}{m}; \alpha_1\right) = u\left(z^{*+} - T(z^{*+}), \frac{z^{*+}}{m}; \alpha_1\right)$. Thus, $\hat{c}(z; m, \alpha_1, \bar{u})$ denotes the consumption level for any income z that keeps utility for type (m, α_1) constant at $\bar{u}$. By construction, $\hat{c}(z^{*-}) = c(z^{*-})$ and $\hat{c}(z^{*+}) = c(z^{*+})$ (where, for ease of notation, we omit $m, \alpha_1, \bar{u}$ as arguments of $\hat{c}(\cdot)$).

Given $\hat{c}(z^{*-}) = c(z^{*-})$ and $\hat{c}(z^{*+}) = c(z^{*+})$, we get $T(z^{*-}) = z^{*-} - \hat{c}(z^{*-})$ and $T(z^{*+}) = z^{*+} - \hat{c}(z^{*+})$. Thus, by the Mean Value Theorem we know for some $z_1 \in (z^{*-}, z^{*+})$ the following holds:

$$\frac{T(z^{*+}) - T(z^{*-})}{z^{*+} - z^{*-}} = \left(\frac{\partial(z - \hat{c}(z))}{\partial z}\right)\Big|_{z=z_1} = 1 + \frac{1}{m}\frac{u_l\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)}{u_c\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)} \tag{41}$$

where we used the implicit function theorem on $u\left(\hat{c}(z), \frac{z}{m}; \alpha_1\right) = \bar{u}$ to get $\frac{\partial \hat{c}(z)}{\partial z} = -\frac{1}{m}\frac{u_l\left(\hat{c}(z), \frac{z}{m}; \alpha_1\right)}{u_c\left(\hat{c}(z), \frac{z}{m}; \alpha_1\right)}$.

Similarly, by the Mean Value Theorem we know for some $z_2 \in (z^{*-}, z^{*+})$ the following holds:

$$\begin{aligned}
\frac{u_l^{*+} z^{*+} - u_l^{*-} z^{*-}}{z^{*+} - z^{*-}} &= \left(\frac{\partial}{\partial z} u_l\left(\hat{c}(z), \frac{z}{m}; \alpha_1\right) z\right)\Big|_{z=z_2} \\
&= \left(u_l\left(\hat{c}(z_2), \frac{z_2}{m}; \alpha_1\right) - u_{cl}\left(\hat{c}(z_2), \frac{z_2}{m}; \alpha_1\right)\frac{u_l\left(\hat{c}(z_2), \frac{z_2}{m}; \alpha_1\right)}{u_c\left(\hat{c}(z_2), \frac{z_2}{m}; \alpha_1\right)}\frac{z_2}{m} + u_{ll}\left(\hat{c}(z_2), \frac{z_2}{m}; \alpha_1\right)\frac{z_2}{m}\right) \\
&\equiv \left(u_l - u_{cl}\frac{u_l}{u_c}\frac{z}{m} + u_{ll}\frac{z}{m}\right)\Big|_{\hat{c}, z=z_2}
\end{aligned} \tag{42}$$

Thus, we can rewrite our optimality condition using Equations 41 and 42, substituting in $T'(\tilde{z}) = 1 + \frac{\frac{1}{n(\tilde{z},\alpha_1)}\tilde{u}_l}{\tilde{u}_c}$, and multiplying both sides by -1:

$$\frac{\left(1 + \frac{1}{m}\frac{u_l\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)}{u_c\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)}\right) u_c^{*-}}{-\left(u_l - u_{cl}\frac{u_l}{u_c}\frac{z}{m} + u_{ll}\frac{z}{m}\right)\Big|_{\hat{c}, z=z_2}} m^2 f(m, \alpha_1) =$$

$$\lim_{(\tilde{z} \to z^{*-})^-} \frac{\left(1 + \frac{1}{n(\tilde{z},\alpha_1)}\frac{\tilde{u}_l}{\tilde{u}_c}\right)\tilde{u}_c}{-\tilde{u}_l + \frac{\tilde{z}}{n(\tilde{z},\alpha_1)}\tilde{u}_{cl}\frac{\tilde{u}_l}{\tilde{u}_c} - \frac{\tilde{z}}{n(\tilde{z},\alpha_1)}\tilde{u}_{ll}} n(\tilde{z}, \alpha_1)^2 f(n(\tilde{z}, \alpha_1), \alpha_1)$$

Evaluating the limit on the RHS, noting $n(z^{*-}; \alpha_1) = m$ and that utility is twice continuously differentiable (so we can pass limits through the derivatives of the utility function) we get:

$$\frac{\left(1 + \frac{1}{m}\frac{u_l\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)}{u_c\left(\hat{c}(z_1), \frac{z_1}{m}; \alpha_1\right)}\right) u_c^{*-}}{\left(-u_l + \frac{z}{m}u_{cl}\frac{u_l}{u_c} - \frac{z}{m}u_{ll}\right)\Big|_{\hat{c}, z=z_2}} m^2 f(m, \alpha_1) = \frac{\left(1 + \frac{1}{m}\frac{u_l^{*-}}{u_c^{*-}}\right) u_c^{*-}}{-u_l^{*-} + \frac{z^{*-}}{m}u_{cl}^{*-}\frac{u_l^{*-}}{u_c^{*-}} - \frac{z^{*-}}{m}u_{ll}^{*-}} m^2 f(m, \alpha_1) \tag{43}$$

Let's first compare the numerators on either side of Equation 43. First, note that by our assumption that $T(z)$ is everywhere increasing, we know that $\frac{T(z^{*+})-T(z^{*-})}{z^{*+}-z^{*-}} = \left(1+\frac{1}{m}\frac{u_l\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)}{u_c\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)}\right) > 0$. By our assumption on convexity of indifference curves, we know that $-\frac{1}{m}\frac{u_l\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)}{u_c\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)} > -\frac{1}{m}\frac{u_l\left(\hat{c}(z^{*-}),\frac{z^{*-}}{m};\alpha_1\right)}{u_c\left(\hat{c}(z^{*-}),\frac{z^{*-}}{m};\alpha_1\right)} = -\frac{1}{m}\frac{u_l\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha_1\right)}{u_c\left(c(z^{*-}),\frac{z^{*-}}{m};\alpha_1\right)}$ as $z_1 > z^{*-}$. Thus, $0 < 1+\frac{1}{m}\frac{u_l\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)}{u_c\left(\hat{c}(z_1),\frac{z_1}{m};\alpha_1\right)} < 1+\frac{1}{m}\frac{u_l^{*-}}{u_c^{*-}}$. Moreover, $u_c^{*-} > 0$. Hence we have that the numerators on the LHS and RHS are both positive and the numerator on the LHS is smaller.

Now let's compare the denominators on either side of Equation 43. We assume that $-u_l+\frac{z}{n}u_{cl}\frac{u_l}{u_c}-\frac{z}{n}u_{ll}$ is increasing in z along each individual's indifference curve so that the denominator of Equation 43 on the LHS is bigger than the denominator of Equation 43 on the RHS as $z_2 > z^{*-}$. Given that $-u_l+\frac{z}{n}u_{cl}\frac{u_l}{u_c}-\frac{z}{n}u_{ll} > 0$ by the SCP (see Appendix A.3.1), we therefore have that both denominators are positive.

Thus, we have that the LHS of Equation 43 is smaller than the RHS of Equation 43 meaning Equation 43 cannot hold. This proves that the tax schedule cannot be differentiable at z^{*-}; an entirely analogous argument can be used to show that the tax schedule cannot be differentiable at z^{*+}. Hence, any optimal tax schedule that is increasing must feature marginal tax rates that change discontinuously at all $z \in \{z_i^{mult}\}$. Finally, to show that the tax schedule must feature marginal tax rates that *increase* discontinuously $\forall\ z \in \{z_i^{mult}\}$, we simply note that if the tax schedule decreased discontinuously at some $z \in \{z_i^{mult}\}$, then no individual would find that z optimal, which contradicts the fact that z is an optimal income for some individual.[69] Hence, the tax schedule must increase discontinuously at all $z \in \{z_i^{mult}\}$. □

A.14 Proof of Proposition 6

Proof. First, we know that there exists a minimum income chosen in society. Specifically, denote $\underline{z}$ as the lowest income chosen in society under the optimal tax schedule, where $\underline{z}$ solves:

$$\underline{z} = \min_{\alpha}\{z^*(\underline{n}_\alpha;\alpha)\}_{\alpha\in A}$$

where $\underline{n}_\alpha = \min_n \text{supp}(f(n|\alpha))$. Moreover, by Lemma 1 we know that any $\underline{n}_\alpha$ locating at $\underline{z}$ must have $\frac{\partial z^*}{\partial n}\big|_{(\underline{n}_\alpha,\alpha)} > 0$.[70] We also know that $H(\tilde{z}|\alpha) = F(n(\tilde{z},\alpha)|\alpha)$, so that $h(\underline{z}) = \sum_{\alpha\in\underline{A}} f(\underline{n}_\alpha,\alpha)\frac{\partial z^*}{\partial n}\big|_{(\underline{n}_\alpha,\alpha)} > 0$, where $\underline{A} = \{\alpha \text{ s.t } z^*(\underline{n}_\alpha;\alpha) = \underline{z}\}$.

[69] An indifference curve diagram in (c,z) space shows that no one locates at kinks where the tax rate decreases.

[70] Given we assume that the lowest income chosen in society under the optimal tax schedule, $\underline{z}$, is not chosen by any individual with multiple optimal incomes, i.e., $\underline{z} \notin \{z_i^{mult}\}$, we know that the tax schedule is twice differentiable at $\underline{z}$ so that $\frac{\partial z^*}{\partial n}\big|_{(\underline{n}_\alpha,\alpha)}$ exists.

Next, consider the following $\tau(z)$ function (with twice continuously differentiable expansions between $[\tilde{z}-d\tilde{z}^2, \tilde{z}]$ and $[\tilde{z}+d\tilde{z}, \tilde{z}+d\tilde{z}+d\tilde{z}^2]$ constructed analogously as in Appendix A.4):

$$\begin{cases} \tau(z) = d\tilde{z} & \text{if } z \leq \tilde{z} - d\tilde{z}^2 \\ \tau(z) = -z + \tilde{z} + d\tilde{z}^2 & \text{if } z \in [\tilde{z}, \tilde{z} + d\tilde{z}] \\ \tau(z) = 0 & \text{if } z \geq \tilde{z} + d\tilde{z} + d\tilde{z}^2 \end{cases}$$

where $\tilde{z}$ and $d\tilde{z}$ are chosen s.t. $\tilde{z} + d\tilde{z} + d\tilde{z}^2 < \min\{z_i^{mult}\}$ and $\tilde{z} + d\tilde{z} + d\tilde{z}^2 < \min\{K_i\}$ (this is possible by Assumption 3 and Assumption 4). Following the same method in the paper of perturbing the optimal tax schedule in the direction of $\tau(z)$, we get the following condition for the optimal tax schedule:

$$\int_{\underline{z}}^{\tilde{z}} (1 - \bar{\omega}(z^*))\, h(z^*)dz^* + \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}\tilde{z}\bar{Z}^c_{\tilde{z}} h(\tilde{z}) - \int_{\underline{z}}^{\tilde{z}} \frac{T'(z^*)}{1 - T'(z^*)}\bar{\eta}_{z^*} h(z^*)dz^* = 0$$

Taking the limit as $\tilde{z} \to \underline{z}$ from the right, we get:

$$\frac{T'(\underline{z})}{1 - T'(\underline{z})}\underline{z}\bar{Z}^c_{\underline{z}} h(\underline{z}) = 0$$

$\frac{\underline{z}\bar{Z}^c_{\underline{z}}}{1-T'(\underline{z})}$ simplifies to $\sum_{\alpha \in \underline{A}} \frac{-u_c\left(\underline{z}-T(\underline{z}), \frac{\underline{z}}{\underline{n}_\alpha}; \alpha\right)}{SOC(\underline{z}, \underline{n}_\alpha, \alpha)} p(\alpha|\tilde{z})$. We know that $-u_c < 0$. Also, $SOC < 0$ $\forall \alpha$ by the fact that $\underline{z} \notin \{z_i^{mult}\}$ and Appendix A.3. Also, we showed above that $h(\underline{z}) > 0$. Thus, in order to satisfy $\frac{T'(\underline{z})}{1-T'(\underline{z})}\underline{z}\bar{Z}^c_{\underline{z}} h(\underline{z}) = T'(\underline{z}) \sum_{\alpha \in \underline{A}} \frac{-u_c\left(\underline{z}-T(\underline{z}), \frac{\underline{z}}{\underline{n}_\alpha}; \alpha\right)}{SOC(\underline{z}, \underline{n}_\alpha, \alpha)} p(\alpha|\tilde{z}) h(\underline{z}) = 0$, it must be the case that the optimal tax schedule has $T'(\underline{z}) = 0$.

□

A.15 Proof of Proposition 7

Proof. By an entirely analogous argument as in the proof of Proposition 6, there exists a maximum income chosen in society $\bar{z}$ with $h(\bar{z}) > 0$.

Again, by assumption $\bar{z} \notin \{z_i^{mult}\}$. From Equation 11, we know that for $\tilde{z} > \max\{z_i^{mult}\}$ and $\tilde{z} > \max\{K_i\}$ (such a maximum exists by Assumption 3 and Assumption 4), the optimal tax schedule must satisfy the following condition:

$$\int_{\tilde{z}}^{\bar{z}} (1 - \bar{\omega}(z^*))\, h(z^*)dz^* - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}\tilde{z}\bar{Z}^c_{\tilde{z}} h(\tilde{z}) - \int_{\tilde{z}}^{\bar{z}} \frac{T'(z^*)}{1 - T'(z^*)}\bar{\eta}_{z^*} h(z^*)dz^* = 0$$

Taking the limit as $\tilde{z} \to \bar{z}$ from the left, we get:

$$-\frac{T'(\bar{z})}{1-T'(\bar{z})}\bar{z}\bar{Z}^c_{\bar{z}}h(\bar{z}) = 0$$

By the same arguments as in the proof to Proposition 6, we know that $\frac{\bar{z}\bar{Z}^c_{\bar{z}}}{1-T'(\bar{z})} > 0$. As we also know that $h(\bar{z}) > 0$, this implies that the optimal tax schedule has $T'(\bar{z}) = 0$.

□

A.16 Deriving Equation 13

We can simplify Equation 12 into a second order differential equation by first using the following relationship, which holds at all points of differentiability of the tax schedule:

$$\begin{aligned} h(\tilde{z}|\alpha) &= f(n(\tilde{z},\alpha)|\alpha)\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}} \\ &= f(n(\tilde{z},\alpha)|\alpha)\left[\frac{u^*_{cc}(1-T'(z^*))^2 + \frac{2}{n}u^*_{cl}(1-T'(z^*)) + \frac{1}{n^2}u^*_{ll} - u^*_c T''(z^*)}{\frac{z^*}{n^2}u^*_{cl}(1-T'(z^*)) + \frac{1}{n^2}u^*_l + \frac{z^*}{n^3}u^*_{ll}}\right]\Big|_{z^*=\tilde{z}} \end{aligned}$$

Substituting this into Equation 12 we get:

$$\sum_{\alpha\in A}\left[-1+\omega(\tilde{z},\alpha) + \frac{T'(\tilde{z})}{1-T'(\tilde{z})}\eta_{\tilde{z},\alpha}\right]\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}} f(n(\tilde{z},\alpha)|\alpha)p(\alpha) - \\ \frac{\partial}{\partial \tilde{z}}\left[\sum_{\alpha\in A}\frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}Z^c_{\tilde{z},\alpha}\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}} f(n(\tilde{z},\alpha)|\alpha)p(\alpha)\right] = 0$$

and plugging in $\tilde{z}Z^c_{\tilde{z},\alpha} = -(1-T'(\tilde{z}))\frac{\tilde{u}_c}{\tilde{u}_{cc}(1-T'(\tilde{z}))^2 + \frac{2}{n}\tilde{u}_{cl}(1-T'(\tilde{z})) + \frac{1}{n^2}\tilde{u}_{ll} - \tilde{u}_c T''(\tilde{z})}$ we get:

$$\sum_{\alpha\in A}\left[-1+\omega(\tilde{z},\alpha) + \frac{T'(\tilde{z})}{1-T'(\tilde{z})}\eta_{\tilde{z},\alpha}\right]\frac{\partial n(z^*,\alpha)}{\partial z^*}\Big|_{z^*=\tilde{z}} f(n(\tilde{z},\alpha)|\alpha)p(\alpha) + \\ \frac{\partial}{\partial \tilde{z}}\left[\sum_{\alpha\in A}\frac{\tilde{u}_c T'(\tilde{z})}{\frac{\tilde{z}}{n^2}\tilde{u}_{cl}(1-T'(\tilde{z})) + \frac{1}{n^2}\tilde{u}_l + \frac{\tilde{z}}{n^3}\tilde{u}_{ll}} f(n(\tilde{z},\alpha)|\alpha)p(\alpha)\right] = 0$$

where $\tilde{u}_c = u_c\left(\tilde{z} - T(\tilde{z}), \frac{\tilde{z}}{n(\tilde{z},\alpha)}; \alpha\right)$ etc.

B Optimal Tax Formulas for Continuously Distributed α

In this section we derive formulas for the optimal tax schedule under the assumption that $F(n,\alpha)$ is twice continuously differentiable in both arguments.

B.1 Government Problem

The government problem is now written as:

$$\begin{aligned}
&\max_{T(z)} \int_A \int_0^\infty W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) dF(n,\alpha) \\
&\text{s.t. } \int_A \int_0^\infty c^*(n,\alpha) dF(n,\alpha) + E \leq \int_A \int_0^\infty z^*(n,\alpha) dF(n,\alpha) \\
&z^*(n,\alpha) \in \operatorname*{argmax}_z u\left(z - T(z), \frac{z}{n}; \alpha\right) \ \forall n, \alpha \\
&c^*(n,\alpha) = z^*(n,\alpha) - T\left(z^*(n,\alpha)\right)
\end{aligned}$$

Without loss of generality, we set $E = 0$. The government Lagrangian is given by:

$$\mathcal{L} = \int_A \int_0^\infty \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T(z^*(n,\alpha))\right] f(n,\alpha) dn d\alpha$$

B.2 Technical Assumptions

We make the following technical assumptions. As in the main text, we assume the (SCP) (Assumption 1). This implies Lemma 1 still holds (i.e., $z^*(n,\alpha)$ is non-decreasing in n $\forall$ α and is increasing in n $\forall$ α whenever $T'(z)$ exists). However, Lemma 2 must be adjusted to the following:

Lemma 4. *If (SCP) holds, the set of individuals with multiple optimal income levels is measure 0.*

Proof. For each α, the set of types n with multiple optima is countable by the proof of Lemma 4. Label them $m_i(\alpha)$ for $i = 1, 2, 3, \ldots$. If a given α only has a finite number k of types n who have multiple optimal incomes, set $m_i(\alpha) = 0$ for $i > k$. Note the graph of the function $m_i(\alpha)$ has measure 0 in $N \times A$ (as the graph of any measurable function is measure 0).[71] Then, the set of individuals with multiple optima is contained in $\cup_{i=1}^\infty m_i(\alpha)$, and the countable union of measure 0 sets is measure 0. □

We continue to assume that Assumption 2 holds, so that, for any α, there exists a

[71] We assume that $m_i(\alpha)$ is a measurable function, which rules out pathological cases.

minimum distance D_1 between the productivity levels of individuals with that α who have multiple optimal income levels. Again, this assumption is needed to rule out pathological settings whereby all individuals with rational productivity levels have multiple optima, for example.

We no longer make Assumption 3 (that there is a minimum difference between the elements of $\{z_i^{mult}\}$) as this no longer seems plausible for continuous α. For instance, if type (n, α_1) has multiple optima at z^{*-}, by continuity arguments, it seems entirely plausible that type $(n, \alpha_1 - \epsilon)$ has a multiple optima at $z^{*-} - \delta$ with $\delta \to 0$ as $\epsilon \to 0$. Instead of Assumption 3, we assume that the set of individuals for whom z^* is one of their multiple optimal income levels is measure zero for all z^* in Assumption 5:

Assumption 5. *Let $A(\tilde{z}, \tilde{z} + d\tilde{z})$ denote the set of types α for whom there is some (n, α) with multiple optimal incomes, one of which is between $\tilde{z}$ and $\tilde{z} + d\tilde{z}$. Then:*

$$\lim_{d\tilde{z} \to 0} \int_{A(\tilde{z}, \tilde{z}+d\tilde{z})} f(\alpha) d\alpha = 0$$

Finally, we make the following assumption on the shape of the optimal tax schedule:

Assumption 6. *$T(z)$ is everywhere twice continuously differentiable.*

Note, we no longer need to allow for the tax schedule to have kink points (as we did in Assumption 4). With discrete α, we needed to allow for kink points in the tax schedule as the income density, $h(z^*)$, discontinuously changes at each $\{z_i^{mult}\}$ as no individual with type α can locate in between any $\{z_i^{*-}(\alpha), z_i^{*+}(\alpha)\}$ by the (SCP). With continuous α, it is still true that no individual of type α type can locate between $\{z_i^{*-}(\alpha), z^{*+}(\alpha)\}$; but, provided that only a measure 0 set of individuals locating at any $\{z_i^{mult}\}$ are multiple optima individuals by Assumption 5, then $h(z^*)$ will evolve continuously so that we no longer need to allow for kinks in the tax schedule.

B.3 All Individuals Have A Unique Optimal Income

We will now derive a differential equation that characterizes the optimal tax schedule. Like in the main text, we first will derive a condition assuming that all agents' have one global optimal income level under the optimal tax schedule. We then relax this assumption and show how this changes the differential equation characterizing the optimal tax schedule.

We know that starting from the optimal tax schedule, the derivative of the government Lagrangian in the direction of $\tau(z)$ must be 0. Thus, the optimal schedule must satisfy the following condition:

$$\frac{\partial}{\partial \mu}\left[\int_A \int_0^\infty \left[W\left(u\left(z^* - T(z^*) - \mu\tau(z^*), \frac{z^*}{n}; \alpha\right); n, \alpha\right) + \lambda(T(z^*) + \mu\tau(z^*))\right] dF(n, \alpha)\right]\Big|_{\mu=0} = 0$$

where we omit that z^* is a function of (n, α) as well as the perturbed schedule $T(\cdot) + \mu\tau(\cdot)$. Because the tax schedule is everywhere differentiable by Assumption 6 we have:

$$\int_A \int_N \left[-W_u(u^*)u_c^*\tau(z^*) + \lambda\left(T'(z^*)\frac{\partial z^*}{\partial \mu}\Big|_{\mu=0} + \tau(z^*)\right)\right] dF(n, \alpha) = 0 \tag{44}$$

where $W_u = \partial W(u; n, \alpha)/\partial u$ and $u^* = u\left(z^* - T\left(z^*\right), \frac{z^*}{n}; \alpha\right)$.

Next, because the SOC holds strictly for all individuals by Lemma 3, we can use Equation 2 to rewrite $\frac{\partial z^*}{\partial \mu}|_{\mu=0}$ in terms of elasticities. We can also rewrite Equation 44 in terms of the optimal income distribution (we can do this because $z^*(n, \alpha)$ is increasing in n by Assumption 6 and Lemma 1):

$$\int_A \int_Z \left[-W_u(u^*)u_c^*\tau(z^*) - \lambda\left(T'(z^*)\left(\frac{Z^c_{z^*,\alpha}z^*}{1 - T'(z^*)}\tau'(z^*) + \frac{\eta_{z^*,\alpha}}{1 - T'(z^*)}\tau(z^*)\right) - \tau(z^*)\right)\right] dH(z^*, \alpha) = 0 \tag{45}$$

where $H(z^*, \alpha) = F(n(z^*, \alpha), \alpha)$, and where $u^*, u_c^*, \frac{\partial z^*}{\partial \mu}|_{\mu=0}$ are now functions of (z^*, α), e.g., $u^* = u\left(z^* - T(z^*), \frac{z^*}{n(z^*,\alpha)}; \alpha\right)$.

We now consider the same $\tau(z)$ function as in the main text (see Appendix A.4 for the definition of $\tau(z)$). Plugging the values of $\tau(z)$ and $\tau'(z)$ into Equation 45, we can use the same logic as in Appendix A.5 to yield Equation 46, a differential equation characterizing the optimal tax schedule:

$$\int_{\tilde{z}}^\infty \left(1 - \bar{\omega}(z^*)\right) h(z^*)dz^* - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})}\tilde{z}\bar{Z}^c_{\tilde{z}}h(\tilde{z}) - \int_{\tilde{z}}^\infty \frac{T'(z^*)}{1 - T'(z^*)}\bar{\eta}_{z^*}h(z^*)dz^* = 0 \tag{46}$$

where $\bar{\omega}(z^*) = \int_A \frac{W_u(u^*)u_c^*}{\lambda} dH(\alpha|z^*)$ denotes the average social welfare weight at income z^*, $\bar{Z}^c_{\tilde{z}} = \int_A Z^c_{\tilde{z},\alpha} dH(\alpha|\tilde{z})$ denotes the average compensated elasticity at income $\tilde{z}$, $\bar{\eta}_{z^*} = \int_A \eta_{z^*,\alpha} dH(\alpha|z^*)$ denotes the average income effect parameter at z^*.

B.4 What if Individuals have Multiple Optimal Incomes?

In deriving Equation 46, we assumed that all agents had one global optimal income level under the optimal tax schedule. However, there is no reason why this assumption need be true. We now proceed to derive the differential equation characterizing the optimal tax schedule allowing for a measure 0 set of agents to have multiple optimal income levels. To do so, first let us assume that there exists at most one n for each α that has multiple optimal income levels under the optimal tax schedule. Denote the productivity levels of these individuals as $m(\alpha)$.[72] Specifically, $m(\alpha)$ satisfies the following indifference condition:

$$u\left(z^{*-}(\alpha)-T(z^{*-}(\alpha)),\frac{z^{*-}(\alpha)}{m(\alpha)};\alpha\right)=u\left(z^{*+}(\alpha)-T(z^{*+}(\alpha)),\frac{z^{*+}(\alpha)}{m(\alpha)};\alpha\right)$$

where z^{*-} and $z^{*+}(\alpha)$ denote $(m(\alpha),\alpha)'s$ minimum and maximum optimal incomes, respectively. Note, we have suppressed that $z^{*-}(\alpha)$, $z^{*+}(\alpha)$, and $m(\alpha)$ are also functions of the tax schedule. Using this notation, we rewrite the government Lagrangian as follows:

$$\mathcal{L}=\int_A\int_0^{m(\alpha)}\left[W\left(u\left(c^*(n,\alpha),\frac{z^*(n,\alpha)}{n};\alpha\right);n,\alpha\right)+\lambda T(z^*(n,\alpha))\right]dF(n,\alpha)+$$
$$\int_A\int_{m(\alpha)}^{\infty}\left[W\left(u\left(c^*(n,\alpha),\frac{z^*(n,\alpha)}{n};\alpha\right);n,\alpha\right)+\lambda T(z^*(n,\alpha))\right]dF(n,\alpha)$$

Now consider perturbing the optimal tax schedule in the direction of $\tau(z)$. Using the same derivation as in Section 3.2, we get:

$$\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}=-\frac{u_c^{*-}(\alpha)\,\tau\left(z^{*-}(\alpha)\right)-u_c^{*+}(\alpha)\,\tau\left(z^{*+}(\alpha)\right)}{u_l^{*-}(\alpha)\,\frac{z^{*-}(\alpha)}{m(\alpha)^2}-u_l^{*+}(\alpha)\,\frac{z^{*+}(\alpha)}{m(\alpha)^2}}\tag{47}$$

where $u_c^{*+}(\alpha)=u_c\left(z^{*+}(\alpha)-T(z^{*+}(\alpha)),\frac{z^{*+}(\alpha)}{m(\alpha)};\alpha\right)$ and $u_l^{*-}(\alpha)=u_l\left(z^{*-}(\alpha)-T(z^{*-}(\alpha)),\frac{z^{*-}(\alpha)}{m(\alpha)};\alpha\right)$, etc.

Next, we use Leibniz's integral rule to take the derivative of the government Lagrangian in the direction of $\tau(z)$, starting from the optimal tax schedule:

$$\int_A\int_0^{\infty}\left[-W_u(u^*)u_c^*\tau(z^*)+\lambda\left(\left.\frac{\partial T(z^*)}{\partial\mu}\right|_{\mu=0}+\tau(z^*)\right)\right]dF(n,\alpha)+$$
$$\int_A\left[W\left(u^{*-}(\alpha)\right)+\lambda T(z^{*-}(\alpha))\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}\right]f(m(\alpha)|\alpha)dF(\alpha)-$$
$$\int_A\left[W\left(u^{*+}(\alpha)\right)+\lambda T(z^{*+}(\alpha))\left.\frac{\partial m(\alpha)}{\partial\mu}\right|_{\mu=0}\right]f(m(\alpha)|\alpha)dF(\alpha)=0$$

Note that in the first term above, we integrate over the entire set of individuals (even those with multiple optima). The value that we assign to $\frac{\partial T(z^*)}{\partial\mu}|_{\mu=0}$ for those with multiple optima is irrelevant because the set with multiple optima is measure 0, so does not affect

[72] If no type α individual has multiple optimal incomes, set $m(\alpha)=0$.

the integral in the first term. Noting that $u^{*-}(\alpha) = u^{*+}(\alpha)\ \forall\ \alpha$, we get:

$$\int_A \int_0^\infty \left[-W_u(u^*)u_c^*\tau(z^*) + \lambda \left(\frac{\partial T(z^*)}{\partial \mu}\Big|_{\mu=0} + \tau(z^*) \right) \right] dF(n,\alpha) + \\ \underbrace{\int_A \lambda \left(T\left(z^{*-}(\alpha)\right) - T\left(z^{*+}(\alpha)\right) \right) \frac{\partial m(\alpha)}{\partial \mu}\Big|_{\mu=0} f(m(\alpha)|\alpha)}_{\text{jumping effects}} dF(\alpha) = 0 \tag{48}$$

Let us now explore the jumping effects in Equation 48 in more detail. We know by Equation 47 that the value of these jumping effects will depend on the tax changes experienced at $z^{*-}(\alpha)$ and $z^{*+}(\alpha)$. We consider the same $\tau(z)$ function as in the main body of the text (see Appendix A.4 for the definition of $\tau(z)$).

As in Section 3.2, if $z^{*+}(\alpha) < \tilde{z} - d\tilde{z}^2$, the jumping effect for these $\alpha's$ will be equal to 0 as as $\tau\left(z^{*+}(\alpha)\right) = \tau\left(z^{*-}(\alpha)\right) = 0$ implying $\frac{\partial m(\alpha)}{\partial \mu}|_{\mu=0} = 0$.

Next consider $z^{*+}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$ and $z^{*-}(\alpha) < \tilde{z} - d\tilde{z}^2$. For these $\alpha's$, the jumping effect in Equation 48 will be equal to:

$$\lambda \left(T\left(z^{*-}(\alpha)\right) - T\left(z^{*+}(\alpha)\right) \right) \frac{u_c^{*+}(\alpha)\, d\tilde{z}}{u_l^{*-}(\alpha) \frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha) \frac{z^{*+}(\alpha)}{m(\alpha)^2}} f\left(m(\alpha)|\alpha\right) \equiv J_1(\alpha)\lambda d\tilde{z}$$

as $\tau\left(z^{*+}(\alpha)\right) = d\tilde{z}$ and $\tau\left(z^{*-}(\alpha)\right) = 0$.

Now consider $z^{*-}(\alpha) > \tilde{z} + d\tilde{z} + d\tilde{z}^2$. For these $\alpha's$, the jumping effect in Equation 48 will be equal to:

$$\lambda \left(T\left(z^{*-}(\alpha)\right) - T\left(z^{*+}(\alpha)\right) \right) \frac{u_c^{*+}(\alpha)\, d\tilde{z} - u_c^{*-}(\alpha)\, d\tilde{z}}{u_l^{*-}(\alpha) \frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha) \frac{z^{*+}(\alpha)}{m(\alpha)^2}} f\left(m(\alpha)|\alpha\right) \equiv J_2(\alpha)\lambda d\tilde{z}$$

as $\tau\left(z^{*+}(\alpha)\right) = \tau\left(z^{*-}(\alpha)\right) = d\tilde{z}$.

However, differently from Section 3.2, we can no longer pick $\tilde{z}$ and $d\tilde{z}$ to ensure that $\{z_i^{mult}\} \notin [\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ because we no longer make Assumption 3. Let $A(\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2)$ denote the set of α for whom at least one of $(m(\alpha), \alpha)'s$ optimal incomes fall in $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$.[73]

Note, from the definition of $\tau(z)$ in Appendix A.4, $\tau(z) \in [0, d\tilde{z} + 2d\tilde{z}^2]$, so that the jumping effect for each of these $\alpha's$ is less than:

$$\lambda \left(T\left(z^{*+}(\alpha)\right) - T\left(z^{*-}(\alpha)\right) \right) \frac{u_c^{*-}(\alpha)\left(d\tilde{z} + 2d\tilde{z}^2\right)}{u_l^{*-}(\alpha) \frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha) \frac{z^{*+}(\alpha)}{m(\alpha)^2}} f(m(\alpha)|\alpha)$$

[73] Note, if we were to relax the assumption that only one n for each α has multiple incomes, we know that by Assumption 2, for sufficiently small $d\tilde{z}$, only one $(m_i(\alpha), \alpha)$ can have a multiple optimal income within $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ for each α.

Thus, the total jumping effect for those with a multiple optima in $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ is less than:

$$\int_{A(\tilde{z}-d\tilde{z}^2,\tilde{z}+d\tilde{z}+d\tilde{z}^2)} \lambda \left(T\left(z^{*+}(\alpha)\right) - T\left(z^{*-}(\alpha)\right)\right) \frac{u_c^{*-}(\alpha)\,(d\tilde{z} + 2d\tilde{z}^2)}{u_l^{*-}(\alpha)\,\frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha)\,\frac{z^{*+}(\alpha)}{m(\alpha)^2}} f(m(\alpha)|\alpha) dF(\alpha)$$
$$< \int_{A(\tilde{z}-d\tilde{z}^2,\tilde{z}+d\tilde{z}+d\tilde{z}^2)} (d\tilde{z} + 2d\tilde{z}^2) B dF(\alpha)$$

where B is a constant that bounds $\lambda \left(T\left(z^{*+}(\alpha)\right) - T\left(z^{*-}(\alpha)\right)\right) \frac{u_c^{*-}(\alpha)}{u_l^{*-}(\alpha)\frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha)\frac{z^{*+}(\alpha)}{m(\alpha)^2}} f(m(\alpha)|\alpha)$ on $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$.[74] Finally, as in Section 3.2 we divide Equation 48 by $\lambda d\tilde{z}$ and taking the limit as $d\tilde{z} \to 0$. The jumping effect for those with multiple optima in $[\tilde{z} - d\tilde{z}^2, \tilde{z} + d\tilde{z} + d\tilde{z}^2]$ goes to 0 by Assumption 5. Rearranging the other terms as in Appendix A.5 we get an analogous optimality condition for the tax schedule:

$$\int_{\tilde{z}}^{\infty} (1 - \bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1 - T'(\tilde{z})} \tilde{z} \bar{Z}_{\tilde{z}}^c h(\tilde{z}) - \int_{\tilde{z}}^{\infty} \frac{T'(z^*)}{1 - T'(z^*)} \bar{\eta}_{z^*} dH(z^*)$$
$$+ \int_A \left[J_1(\alpha) \mathbb{1}\left(z^{*-}(\alpha) < \tilde{z} < z^{*+}(\alpha)\right) + J_2(\alpha) \mathbb{1}\left(z^{*-}(\alpha) > \tilde{z}\right)\right] f(\alpha) d\alpha = 0 \qquad (49)$$

Equation 49 gives us a condition that the optimal tax schedule must satisfy at all income levels under the assumption that there is at most one type n for each α with multiple optima.

Finally, we can relax the assumption that there exists at most one n for each α with multiple optima. As in Section 3.2, we allow there to exist a countable number of individuals with multiple optima for each α. Denote $m_i(\alpha)$ as the i^{th} productivity level with multiple optimal incomes, and denote their minimum optimal income as $z_i^{*-}(\alpha)$ and their maximum optimal income as $z_i^{*+}(\alpha)$. Denote the number of individuals with multiple incomes for a given α as $M(\alpha)$ (which can also be countably infinite or zero). Given this new notation, we can write the government's Lagrangian as:[75]

$$\mathcal{L} = \int_A \int_0^{m_1(\alpha)} \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T(z^*(n,\alpha))\right] dF(n,\alpha) +$$
$$\int_A \sum_{i=1}^{M(\alpha)-1} \int_{m_i(\alpha)}^{m_{i+1}(\alpha)} \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T(z^*(n,\alpha))\right] dF(n,\alpha) +$$
$$\int_A \int_{m_{M(\alpha)}(\alpha)}^{\infty} \left[W\left(u\left(c^*(n,\alpha), \frac{z^*(n,\alpha)}{n}; \alpha\right); n, \alpha\right) + \lambda T(z^*(n,\alpha))\right] dF(n,\alpha)$$

[74] We assume such a bound exists. Given that $u_l^{*-}(\alpha)\frac{z^{*-}(\alpha)}{m(\alpha)^2} - u_l^{*+}(\alpha)\frac{z^{*+}(\alpha)}{m(\alpha)^2} > 0\ \forall (m(\alpha), \alpha)$ by SCP (see Appendix A.13), if we assume utility is continuous in α (as well as in n) and the support of $F(n,\alpha)$ is closed and bounded, the extreme value theorem gives us that such a bound exists.

[75] We've used Assumption 2 to ensure that the set $\{m_i(\alpha)\}$ can be totally ordered using the usual relation $<$, so that we can write out the Lagrangian as a sum over integrals with endpoints in $\{m_i(\alpha)\}$.

Using identical logic as before, we can augment Equation 49 as:

$$\int_{\tilde{z}}^{\infty} (1-\bar{\omega}(z^*))\, dH(z^*) - \frac{T'(\tilde{z})}{1-T'(\tilde{z})}\tilde{z}\bar{Z}^c_{\tilde{z}}h(\tilde{z}) - \int_{\tilde{z}}^{\infty} \frac{T'(z^*)}{1-T'(z^*)}\bar{\eta}_{z^*}dH(z^*)$$
$$\int_A \sum_{i=1}^{M(\alpha)} \left[J_{1i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha) < \tilde{z} < z_i^{*+}(\alpha)\right) + J_{2i}(\alpha)\mathbb{1}\left(z_i^{*-}(\alpha) > \tilde{z}\right)\right] f(\alpha)d\alpha = 0 \tag{50}$$

Equation 50 gives us a differential equation the optimal tax schedule must satisfy at all income levels.

C Simulation Appendix

C.1 Step-by-Step Simulation Procedure

Below we discuss our simulation procedure when we allow for the possibility that one individual has multiple optima under the optimal tax schedule (note, this is an endogenous assumption and needs to be checked). The procedure can be augmented to allow for more individuals with multiple optimal incomes. This procedure is written for the utility function and government social welfare function described in Section 5.2.

First, set values of α_1,α_2 (note, under linear taxes, $Z_i^c = \frac{1}{\alpha_i}$), where, WLOG, $\alpha_1 < \alpha_2$. Set primitive distributions: $f(n|\alpha_1)$ for $n \in N_{\alpha_1}$ and $f(n|\alpha_2)$ for $n \in N_{\alpha_2}$ where N_{α_i} are closed, bounded sets. Let $p(\alpha_1), p(\alpha_2)$ reflect proportions of type 1 and 2 in society. Note, for simplicity, we choose the minimum skill for type 2, $\underline{n}_{\alpha_2}$ to satisfy $\underline{n}_{\alpha_2}^{\frac{1+\alpha_2}{\alpha_2}} = \underline{n}_{\alpha_1}^{\frac{1+\alpha_1}{\alpha_1}}$, and the maximum skill for type 2, $\overline{n}_{\alpha_2}$, to satisfy $\overline{n}_{\alpha_2}^{\frac{1+\alpha_2}{\alpha_2}} = \overline{n}_{\alpha_1}^{\frac{1+\alpha_1}{\alpha_1}}$ so that both type 1 and type 2 locate at the minimum and maximum incomes chosen in society under the optimal tax schedule, which features 0 marginal rates at the top and bottom - see Propositions 6 and 7, respectively. We assume some individual with type α_1 has two optimal income levels.[76] We check afterwards that no α_2 individual has multiple optimal incomes.

1. Choose initial values for $\left[\lambda, T(\underline{z}), (n_l, \alpha_1), (m, \alpha_1), T'_{j1}, T'_{j2}\right]$, where T'_{j1}, T'_{j2} denote the sizes of the two jumps in the marginal tax rate schedule, (m, α_1) denotes our individual with multiple optima, and (n_l, α_1) denotes the individual who picks the same income level as the individual with multiple optima, (m, α_1), and who's FOC is satisfied (from the left) at this income level, i.e., (n_l, α_1) has his FOC satisfied from the left at z^{*-} where z^{*-} denotes the minimum income chosen by our multiple

[76] Indifference curves for type α_1 individuals are less steep than for type α_2 type individuals; one can see from an indifference curve diagram that individuals with less steep indifference curves will be more likely to have multiple optimal incomes than individuals with steeper indifference curves.

optima individual (m, α_1). We assume exogenous government expenditures, E, are 0.

2. Determine initial values:

   (a) Set $T'(\underline{z}) = 0$, where $\underline{z}$ solves:

   $$\underline{z} = \min\left(\underline{n}_{\alpha_1}^{\frac{1+\alpha_1}{\alpha_1}}, \underline{n}_{\alpha_2}^{\frac{1+\alpha_2}{\alpha_2}}\right)$$

   where $\underline{n}_{\alpha_i} = \min N_{\alpha_i}$. This expression comes from substituting in $T'(\underline{z}) = 0$ into $(\underline{n}_{\alpha_1}, \alpha_1)$ and $(\underline{n}_{\alpha_2}, \alpha_2)$ FOCs. Note, by our choice of N_{α_1} and N_{α_2}, we know that $\underline{n}_{\alpha_1}^{\frac{1+\alpha_1}{\alpha_1}} = \underline{n}_{\alpha_2}^{\frac{1+\alpha_2}{\alpha_2}}$.

   (b) Determine optimal utility at $\underline{z}$, $u(\underline{z}; \alpha_i)$, using following expression:

   $$u(z; \alpha_i) = z + T(z) - \frac{\left(\frac{z}{\underline{n}_{\alpha_i}}\right)^{1+\alpha_i}}{1 + \alpha_i}$$

   (c) Save vector of starting values: $[\underline{z}, Y(\underline{z})] = [\underline{z}, T'(\underline{z}), u(\underline{z}; \alpha_1), u(\underline{z}, \alpha_2)]$

3. Determine $Y'(z) = [T''(z), u'(z; \alpha_1), u'(z, \alpha_2)]$:

   (a) For $i = 1, 2$, calculate the n_i for type α_i whose FOC is satisfied at z:

   $$n_i(z; \alpha_i) = \left(\frac{z^{\alpha_i}}{1 - T'(z)}\right)^{\frac{1}{1+\alpha_i}}$$

   (b) For $i = 1, 2$, calculate $f(n_i|\alpha_i)$

   (c) If $n(z; \alpha_1) = n_l$:[77]

      i. Change the marginal tax rate at z to $T'(z) + T'_{j1}$.

      ii. Save the value of this income level as z^{*-}.

      iii. Calculate the utility for our multiple optima individual:[78]

      $$u(z^{*-}; m, \alpha_1) = z^{*-} - T(z^{*-}) - \frac{\left(\frac{z^{*-}}{m}\right)^{1+\alpha_1}}{1 + \alpha_1}$$

      iv. Restart at the beginning of Step 3.

   (d) If $z > z^{*-}$ and $u(z; m, \alpha_1) < u(z^{*-}; m, \alpha_1)$:

      i. Change $f(n_1|\alpha_1) = 0$ (because no α_1 type can locate in (z^{*-}, z^{*+}))

[77] Technically, we check that $n(z; \alpha_1) \geq n_l$ *and* $n(z - \epsilon; \alpha_1) < n_l$.

[78] Denoting the skill level of individual who's FOC is satisfied from the right at income z^{*-} as n_r, we know our indifferent individual's skill level m must satisfy $n_l \leq m \leq n_r$.

(e) If $u(z, m, \alpha_1) = u(z^{*-}; m, \alpha_1)$:[79]

i. Change the marginal tax rate at z to be $T'(z) = T'(z) + T'_{j2}$

ii. Restart at the beginning of Step 3.

(f) Calculate $\omega(z; \alpha_i)$ using following equation:

$$\omega(z; \alpha_i) = \frac{1}{u(z; \alpha_i)\lambda} = \frac{1}{\lambda \left(z + T(z) - \frac{\left(\frac{z}{n(z;\alpha_i)}\right)^{1+\alpha_i}}{1+\alpha_i} \right)}$$

(g) Solve for $T''(z)$, $\frac{\partial n(z;\alpha_1)}{\partial z}$, and $\frac{\partial n(z;\alpha_2)}{\partial z}$ using Equation 14 and the following equation (which holds for $i = 1, 2$):

$$\frac{\partial n(z; \alpha_i)}{\partial z} = \frac{\alpha_i}{1 + \alpha_i} (z(1 - T'(z)))^{\frac{-1}{1+\alpha_i}} + \frac{1}{1 + \alpha_i} z^{\frac{\alpha_i}{1+\alpha_i}} (1 - T'(z))^{\frac{-2-\alpha_i}{1+\alpha_i}} T''(z)$$

(h) Calculate $\frac{\partial u(z;\alpha_i)}{\partial z}$ using the following equation:[80]

$$\frac{\partial u(z; \alpha_i)}{\partial z} = \frac{z^{1+\alpha_i}}{n(z; \alpha_i)^{2+\alpha_i}} \frac{\partial n(z; \alpha_i)}{\partial z}$$

(i) Save $Y'(z) = [T''(z), \frac{\partial u(z;\alpha_1)}{\partial z}, \frac{\partial u(z;\alpha_2)}{\partial z}]$

(j) Save $f(n_1(z; \alpha_1)|\alpha_1)$ and $f(n_2(z; \alpha_2)|\alpha_2)$

4. Determine $Y(z + \epsilon)$ using a first-order Taylor Series expansion:

$$Y(z + \epsilon) = Y(z) + \epsilon Y'(z)$$

5. Repeat steps 3 and 4 until $z = \overline{z}$, the highest income chosen by any type.

6. Under the tax schedule that solves the differential equation for given initial values, calculate:

$$R = \sum_{i=1}^{2} p(\alpha_i) \int_{\underline{n}_{\alpha_i}}^{\overline{n}_{\alpha_i}} T(z(n; \alpha_i)) f(n|\alpha_i) dn$$

where $z(n; \alpha_i)$ solves $z = (n^{1+\alpha_i}(1 - T'(z)))^{\frac{1}{\alpha_i}}$.[81]

7. Search over values of $T(\underline{z})$ until $R = 0$ so that the government budget constraint is satisfied.

8. Search to find λ, (n_l, α_1), (m, α_1), T'_{j1}, T'_{j2} that maximize welfare.

[79]Technically, we check that $u(z; m, \alpha_1) \geq u(z^{*-}; m, \alpha_1)$ and $u(z - \epsilon; m, \alpha_1) < u(z^{*-}; m, \alpha_1)$.

[80]This expression comes from applying the envelope theorem and then doing a change of variables from n to z.

[81]Note, we account for the fact that there is bunching at kink points when calculating R.

9. Check that z assigned to each person (n, α) maximizes utility for that type.

C.2 Optimal Consumption Schedules

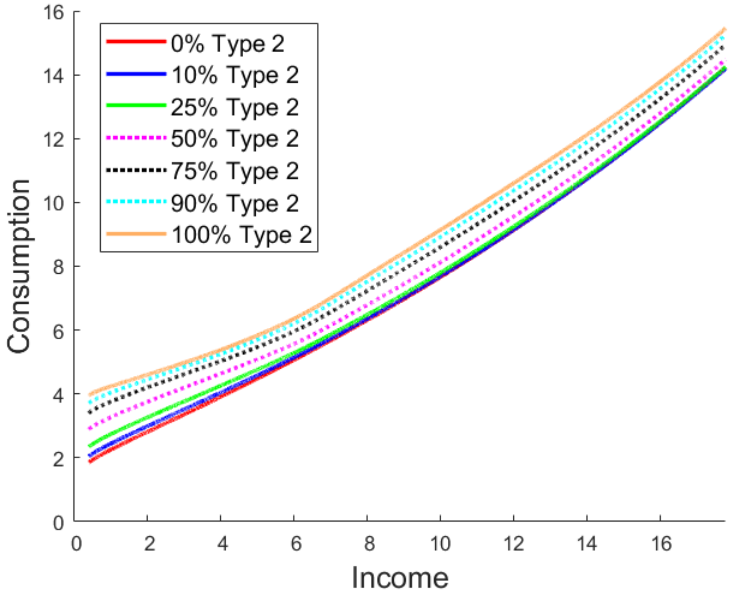

Figure 10: Optimal Consumption Schedules for Various Percentages of Type 2 Individuals